Some definite integrals of the logarithm, ln(x), with rational functions, using complex contour integration and infinite series

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In this document I describe a hobby interest I have enjoyed, working out some definite integrals involving $\ln x$ ($\log_e x$) combined with rational functions of polynomials. I am sharing this over the Internet in the hope that it might help other students gain confidence with complex integration around contours in the complex plane. In some cases I have been unable to find a contour capable of giving an evaluation in closed form, and I suspect that such a contour may not exist; I have therefore tried to perform a typical such integral using summation of infinite series. The methods used here were all known well before 1850; there is nothing here for the modern researcher. Moreover, most of the integrals here have been studied by other authors and several are in text books. I cite these references and point out some errors in books. I make little claim for originality of conclusion, though all the working and comments here are my own.

My exploration of this topic has been just that – a personal journey through an area of maths with many dead ends, moments of bafflement, interesting diversions and a few great views on the way. Accordingly, my account is not presented as a formal maths paper for a refereed journal, but as a rambling narrative in which I invite the reader to share. In places I conjecture some general results; perhaps other enthusiasts will either prove or disprove them.

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1. The initial challenge

I got the idea for investigating these integrals from a puzzle question set in the 2005 William Lowell Putnam maths competition, organised by the Mathematics Association of America:

$$Evaluate \int_0^1 \frac{\ln(x+1)}{x^2+1}\,dx.$$

The integrand is well behaved, but it does not have a 'primitive' – that is, a function which, when differentiated with respect to x, gives the integrand. Numerical integration gives 0·272198. Following a hunch motivated by the numerator and denominator of the integrand, I identified this with $\frac{\pi}{8}\ln 2 = 0{\cdot}272198261288\cdots$. So we have an answer, but what about the proof? I could have looked up the answer on the Putnam competition web site, but instead chose to have a go myself – and I have not yet looked up the official answer. My personal search for a satisfactory proof led me to consider this and related integrals. I hope you share in some of my enjoyment.

In my first attempt I converted the integrand into infinite series, integrated term by term, and showed that the result could be rearranged to look like the product of two series, one for $\frac{\pi}{4}$ and the other for $\ln 2$. This is described in the Appendix. This method requires that the series involved be *uniformly* convergent, since only then can they be manipulated and still converge in a well behaved manner.

My explorations of this topic were well advanced when I came across a book, published in 2004 by C.U.P, with the wonderful title *Irresistible Integrals* by George Boros and Victor Moll. On page 243 they give a short, clever and elegant proof of the Putnam challenge integral, commenting that it was originally evaluated in 1844 by Serret of the Serret-Frenet differential geometry formula. (Why is it that, when you have toiled over some maths for many hours or even days, someone points out a three line solution!?) Here is the proof based on that in Boros and Moll's book. It uses only real variables and the symmetry of trigonometric functions.

Make the change of variable $x = \tan\theta$, $dx = d\theta/\cos^2\theta = (1+\tan^2\theta)\,d\theta$ to obtain

$$\int_0^1 \frac{\ln(x+1)}{x^2+1}dx = \int_0^{\pi/4} \ln(\tan\theta\,+\,1)\,d\theta.$$

Now use $\frac{1}{\sqrt{2}}(\sin\theta + \cos\theta) = \sin(\theta + \pi/4)$:

$$\int_0^1 \frac{\ln(x+1)}{x^2+1}dx = \int_0^{\pi/4} \left[\ln\sqrt{2} + \ln\,\sin(\theta + \tfrac{\pi}{4}) - \ln\,\cos\theta\right]\,d\theta = \tfrac{1}{8}\pi\ln 2$$

because the two log-trigonometric integrals cancel by virtue of their reflection symmetry in the line $\theta = \pi/8$. Isn't that neat! It relies, however, on symmetries not possessed by similar integrals and so probably cannot be widely generalised.

2. Background on complex integration of functions involving $\ln x$

2.1 The meaning of evaluating an integral

At the risk of being tedious, may I first share some philosophical points about what we mean by 'evaluating an integral algebraically'With indefinite integrals it means finding a 'primitive'; that is, some combination of elementary functions (or at least well understood and tabulated functions such as Bessel functions) which, when differentiated, gives the integrand in question. In school when we start learning calculus, the integration exercises have been selected to have primitives which can be discovered by standard techniques such as integration by substitution or by parts. Yet it is well known that these classroom problems are the exceptions; *most* indefinite integrals composed of combinations of polynomials and elementary function do *not* have a primitive. Moreover, as far as I know, there is no way of telling in advance which integrand will have a primitive. For example,

$$\int \frac{\ln x}{x}\,dx = \int \ln x\,d(\ln x) = \tfrac{1}{2}\ln^2 x$$

in readily integrated, but

$$\int \frac{x}{\ln x}\, dx = Ei(2\,\ln x)$$

does not have a primitive in terms of $\ln x$; instead it requires the special function, $Ei(x)$, which is the tabulated Exponential Integral. A simple change to

$$\int \frac{x}{\ln(x+1)}\, dx$$

and the integral does not have a primitive even in terms of the Exponential Integral. It is as if the set of functions which can be made from finite combinations of polynomials and elementary tabulated functions, infinite though it is, is not sufficient to provide a primitive for most of the integrands which are themselves composed of polynomials and elementary functions. I see an analogy here with the algebraic numbers and the real numbers, in that the algebraics, though infinite in number, are not a complete set – there are an infinity of transcendental numbers which inhabit the spaces between every pair of adjacent algebraic numbers.

Much the same can be said of the evaluation of definite integrals. Some will turn out to have values which we can express in terms of well known constants such as π or $\ln 2$, but most will not — they are simply themselves and have their own value, not shared with other integrals. Therefore much of this paper is about trying to find a few definite integrals involving the logarithm which can be evaluated in terms of well documented constants. One might question whether this gives much more insight than merely knowing the numerical value from quadrature, considered that numerical integration of many smooth functions can be now performed to 20 or more significant figures. For instance, does knowing that the value of the Putnam challenge integral is $\pi \ln 2/8$ give us any more understanding than knowing that it has a value very close to $0 \cdot 272198261288$? I think the answer is *Yes*. It tells us something about the anatomy of the integral and the ubiquitous natures of π and $\ln 2$. At first sight these two constants seem to have little in common. However they can be represented as integrals and as infinite series which show a similar structure:

$$\begin{aligned} \arctan 1 \equiv \tan^{-1} 1 = \tfrac{\pi}{4} = 1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} - \cdots &= \int_0^1 \frac{dx}{x^2+1} \\ \ln 2 = 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \cdots &= 2\int_0^1 \frac{x\,dx}{x^2+1} \end{aligned} .$$

In this paper we will meet π and $\ln 2$ many times.

But that's enough philosophy for now.

2.2 Structure of the complex logarithm

I presume the reader to be acquainted with, if not expert in, complex integration for evaluation of definite integrals. Several text books give integrals involving $\ln x$ and I give a list of those I have found helpful in the §4. We will be using the residue theorem: that the integral round any closed contour of an analytic function is equal to $2\pi\imath$ times the sum of residues at the poles enclosed by the contour.

May I next ask you also to recall the properties of the complex logarithm. If $z = x + \imath y$ is expressed in exponential form $\sqrt{(x^2+y^2)}\; e^{\imath\theta}$ where $\tan\theta = y/x$, the logarithm of z is

$$\ln z = \ln \sqrt{x^2+y^2} + \imath\theta = \frac{1}{2}\ln(x^2+y^2) + \imath \tan^{-1} y/x.$$

The real part $\to -\infty$ as $x^2 + y^2 \to 0$, 0 being a singular point about which $\ln z$ does not have a Laurent expansion. The arctangent in the imaginary part is infinitely multivalued such that each clockwise circuit around $z = 0$ increases $\Im \ln z$ by 2π, and we say that the path has moved onto another branch of the logarithm. You may picture the logarithm as having a helical structure, like a multi-storey car park; then circling round by a full turn takes you to the next pitch of the helix (the next floor of the car park). Since any closed loop contour of integration must remain on one branch (one floor of the car park), it is necessary to specify a 'branch cut' as a boundary not to be crossed. This cut will run from the branch point at $z = 0$ out to infinity. We are free to choose its direction. Indeed there is no *a priori* reason for the cut to be a straight line, though it is common to cut along the x axis either from $-\infty$ to 0, or from 0 to $+\infty$. Across this cut the value of $\Im \ln z$ jumps discontinuously by 2π.

With this background information, we can evaluate the Putnam challenge integral.

3. The Putnam challenge integral by contour integration: $\displaystyle\int_0^1 \frac{\ln(x+1)}{x^2+1}\,dx$

3.1 Contour evaluation of $\int_0^1 \frac{\ln(x+1)}{x^2+1}\,dx$

Step 1: Consider the singularities of the integrand $\oint \frac{\ln(z+1)}{z^2+1}dz$. $\ln(z+1)$ goes to $-\infty$ as $z \to -1$, and there are two poles, each of order 1, at $z = \pm\imath$, where the denominator goes to zero.

Step 2: Choose a suitable contour around which to integrate, avoiding the singularity in $\ln z$. See Figure 3.1 for my choice. Since the contour does not enclose either pole, the integral round the circuit is zero.

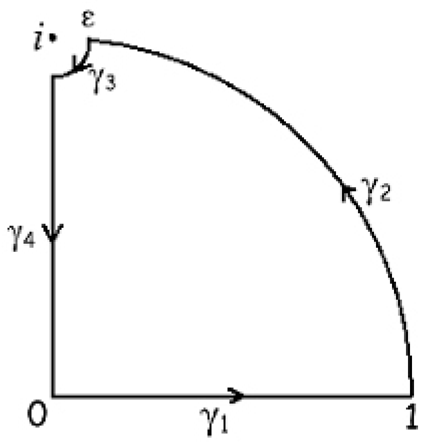

Figure 3.1: Contour in complex plane for $\oint \frac{\ln(z+1)}{z^2+1}dz$.

Step 3: Parameterise each segment of the contour with a real variable:

On $\gamma_1 : z = x$ and $dz = dx$. $I = \int_{\gamma_1}$ is the required integral.

On $\gamma_2 : z = e^{\imath\theta}$ and $dz = \imath e^{\imath\theta}d\theta$.

On $\gamma_3 : z = \varepsilon e^{\imath\theta} + \imath$ and $dz = \imath\varepsilon e^{\imath\theta}d\theta$. ε is small and we will eventually take the limit $\varepsilon \to 0$.

On $\gamma_4 : z = \imath t$ and $dz = \imath dt$, t real.

Step 4: Evaluate each contribution.

On γ_2 :

$$\int_{\gamma_2} = \lim_{\varepsilon\to 0}\int_0^{\frac{\pi}{2}-\varepsilon} \frac{\ln(e^{\imath\theta}+1)}{e^{2\imath\theta}+1}\imath e^{\imath\theta}\ d\theta \ = \ \int_0^{\frac{\pi}{2}} \frac{\ln(e^{\imath\theta}+1)}{e^{2\imath\theta}+1}.\imath e^{\imath\theta}\ d\theta$$

Now $\ln z = \ln|z| + \imath \arg z = \frac{1}{2}\ln|z|^2 + \imath\arg z$ and $|e^{\imath\theta}+1|^2 = 2+2\cos\theta$, so

$$\ln(e^{\imath\theta}+1) = \tfrac{1}{2}\ln(2+2\cos\theta) + \imath\tan^{-1}\left(\frac{\sin\theta}{1+\cos\theta}\right)$$

Now use the trigonometric identity

$$\tan\frac{\theta}{2} = \frac{\sin\theta}{1+\cos\theta} = \frac{1-\cos\theta}{\sin\theta} \qquad Eq.3.1$$

to obtain

$$\ln(e^{\imath\theta}+1) = \frac{1}{2}\ln(2+2\cos\theta) + \frac{\imath\theta}{2}.$$

Incidentally, Figure 3.2 shows an elegant geometrical proof of the above trig identity, using the theorem that the angle subtended at the centre of a circle is twice the angle subtended at the circumference.

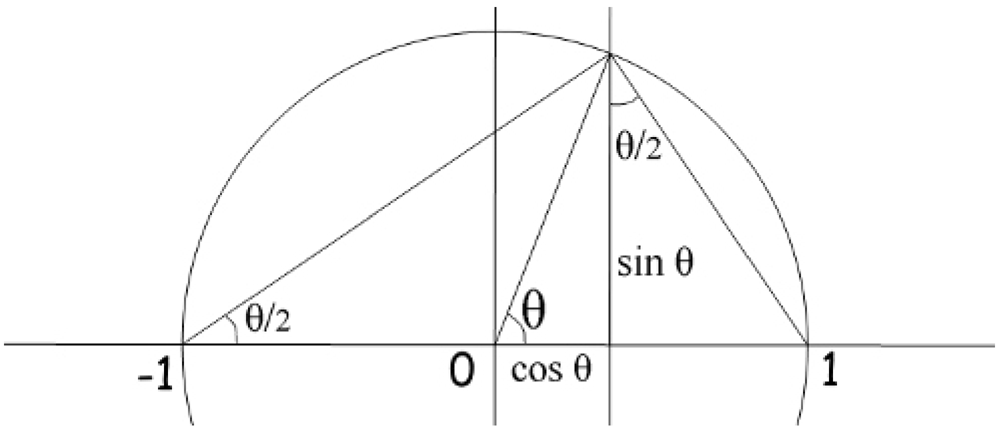

Figure 3.2: Geometrical proof of a half angle trig. identity

Also

$$\frac{e^{\imath\theta}}{e^{2\imath\theta}+1} = \frac{1}{e^{\imath\theta}+e^{-\imath\theta}} = \frac{1}{2\cos\theta}.$$

Collecting this together

$$\int_{\gamma_2} = \frac{\imath}{4}\int_0^{\frac{\pi}{2}} \frac{\ln(2+2\cos\theta)}{\cos\theta}d\theta - \int_0^{\frac{\pi}{2}} \frac{\theta}{4\cos\theta}d\theta \qquad \int_{\gamma_2}$$

On γ_3 :

$$\begin{aligned}
\int_{\gamma_3} &= \lim_{\varepsilon\to 0}\int_0^{-\frac{\pi}{2}} \frac{\ln(\varepsilon e^{\imath\theta}+\imath+1)}{\varepsilon^2 e^{2\imath\theta}+2\imath\varepsilon e^{\imath\theta}}\,\imath\varepsilon e^{\imath\theta}\,d\theta \\
&= \lim_{\varepsilon\to 0}\int_0^{-\frac{\pi}{2}} \frac{\ln(\varepsilon e^{\imath\theta}+\imath+1)}{-\imath\varepsilon e^{2\imath\theta}+2}\,d\theta \\
&= \int_0^{-\frac{\pi}{2}} \frac{\ln(1+\imath)}{2}d\theta \\
&= \frac{1}{2}\left(\frac{1}{2}\ln 2+\imath\frac{\pi}{4}\right)\int_0^{-\frac{\pi}{2}} d\theta \\
\int_{\gamma_3} &= -\frac{\pi}{8}\ln 2-\imath\frac{\pi^2}{16}
\end{aligned}$$

On γ_4

$$\lim_{\varepsilon\to 0}\int_{1-\varepsilon}^0 \frac{\ln(\imath t+1)}{-t^2+1}\,\imath dt = \int_0^1 \frac{\frac{-\imath}{2}\ln(t^2+1)+\tan^{-1}t}{1-t^2}\,dt \qquad \int_{\gamma_4}$$

Step 5: Equate real and imaginary parts to zero, since the integral round the total contour is zero. The real parts are

$$I-\int_0^{\frac{\pi}{2}}\frac{\theta}{4\cos\theta}d\theta-\frac{\pi}{8}\ln 2+\int_0^1\frac{\tan^{-1}t}{1-t^2}dt=0$$

In this, if we can show that the contributions from γ_2 and γ_4 respectively are equal and cancel, we have established that $I=\frac{\pi}{8}\ln 2$, the required result. In fact this equality of integrals can be shown using the substitution $t=\tan\frac{\theta}{2}$. Together with the trigonometric identity Eq. 3.1 above we find

$$\cos\theta=\frac{1-t^2}{1+t^2}\;, \qquad dt=\frac{1}{2}(1+t^2)\,d\theta.$$

Hence

$$\lim_{\eta\to 0}\int_0^{\frac{\pi}{2}-\eta}\frac{\theta}{4\cos\theta}\,d\theta=\lim_{\varepsilon\to 0}\int_0^{1-\varepsilon}\frac{\tan^{-1}t}{2}\left(\frac{1+t^2}{1-t^2}\right)\frac{2}{(1+t^2)}\,dt=\lim_{\varepsilon\to 0}\int_0^{1-\varepsilon}\frac{\tan^{-1}t}{1-t^2}\,dt, \qquad Eq.3.2$$

I have reintroduced the limiting process here because the integrals diverge. (It would be possible to obtain a Cauchy principal value if the limits were symmetric about $\pi/2$ or 1 respectively, since the divergence towards $+\infty$ on the lesser side would be balanced by the divergence towards $-\infty$ on the greater side. But at $\pi/2$ or 1 itself the integral diverges.)

Accepting this cancellation between γ_2 and γ_4 we may conclude from the real part of $\oint\frac{\ln(z+1)}{z^2+1}dz$ that the Putnam competition integral evaluates to $\frac{\pi}{8}\ln 2$. *Q.E.D.*

3.2 Other integrals resulting from the above contour integration

With the substitution $x=u/a$ we readily obtain the more general result

$$\int_0^a\frac{\ln(x+a)}{x^2+a^2}dx=\frac{\pi}{8a}\ln(2a^2), \qquad a\neq 0.$$

$$\text{since} \quad \int_0^a \frac{\ln(x+a)}{x^2+a^2}dx = \int_0^1 \frac{\ln(x+1)}{x^2+1}dx + \frac{\pi}{4}\ln a.$$

Considering now the imaginary parts, the contributions from the four segments of the contour add as follows:

$$0 + \frac{1}{4}\int_0^{\frac{\pi}{2}} \frac{\ln(2+2\cos\theta)}{\cos\theta}d\theta - \frac{\pi^2}{16} - \frac{1}{2}\int_0^1 \frac{\ln(t^2+1)}{1-t^2}dt = 0$$

It is sensible to see how the integrals contributed by γ_2 and γ_4 will combine, so I will again use the substitution $t = \tan\frac{\theta}{2}$ to convert the integral $d\theta$ to an integral dt:

$$\begin{aligned}
\frac{1}{4}\int_0^{\frac{\pi}{2}} \frac{\ln(2+2\cos\theta)}{\cos\theta}d\theta &= \frac{1}{4}\int_0^1 \ln\left[2+2\frac{(1-t^2)}{1+t^2}\right]\left(\frac{1+t^2}{1-t^2}\right)\left(\frac{2.dt}{1+t^2}\right) \\
&= \frac{1}{2}\int_0^1 \ln\left[\frac{4}{1+t^2}\right]\frac{dt}{1-t^2} \\
&= \int_0^1 \frac{\ln 2}{1-t^2}dt - \frac{1}{2}\int_0^1 \frac{\ln(1+t^2)}{1-t^2}\,dt
\end{aligned}$$

This is one of those remarkable examples where, individually, each of these integrals diverges but their difference is finite. The first, in fact, is $\ln 2 \lim_{t\to 1} \tanh^{-1} t$. The second is identical to the contribution from γ_4. Adding these gives

$$\int_0^1 \frac{\ln 2}{1-t^2}dt - \int_0^1 \frac{\ln(1+t^2)}{1-t^2}dt = \frac{\pi^2}{16}.$$

Using the properties of logs these can be recombined into

$$\int_0^1 \frac{\ln[\frac{1}{2}(1+t^2)]}{1-t^2}dt = -\frac{\pi^2}{16} = -0\cdot 61685\,. \qquad Eq.3.3$$

which is an addition to our collection of integrals. I have confirmed this result by numerical integration. This can be put into two other tidy forms. First make the substitution $2/(1+t^2) = u, 1-t^2 = 2-2/u, dt = -du/(tu^2)$:

$$\int_0^1 \frac{\ln[\frac{1}{2}(1+t^2)]}{1-t^2}dt = \int_2^1 \frac{\ln(1/u)}{2-2/u}\left(\frac{-du}{tu^2}\right) = \int_1^2 \frac{\ln u}{2(u-1)\sqrt{2u-u^2}}\,du = -\frac{\pi^2}{16}\,.$$

Now let $u = 1+x$, $2u - u^2 = 1 - x^2$ to obtain

$$\int_0^1 \frac{\ln(x+1)}{x\sqrt{1-x^2}}dx = \frac{\pi^2}{8}\,. \qquad Eq.3.4$$

A trigonometric form of this can be obtained by the further substitution $x = \sin\theta$:

$$\int_0^{\pi/2} \frac{\ln(1+\sin\theta)}{\sin\theta}d\theta = \frac{\pi^2}{8} = 1\cdot 2337\,. \qquad Eq.3.5$$

That concludes my evaluation of integrals arising from the imaginary part of the Putnam integral.

4. Other integrals of logarithms with rational functions given in text books

I have found these books helpful on the techniques of complex integration, and they contain an interesting collection of integrals involving $\ln x$.

4.1 Murray Spiegel : 'Theory and Problems of Complex Variables' in Schaum Outline Series. (My copy is the 1964 edition.) In Chapter 7 page 187 Spiegel does

$$\int_0^\infty \frac{\ln(x^2+1)}{x^2+1}dx = \pi \ln 2$$

as a fully worked example. Similarly on page 193 he does the pair

$$\int_0^\infty \frac{(\ln x)^2}{x^2+1}dx = \frac{\pi^3}{8} \quad , \qquad \int_0^\infty \frac{\ln x}{x^2+1}dx = 0.$$

Note how these two arise from the real and imaginary parts of a complex expression. There are further log integrals set as exercises on page 198, questions 87, 88, 91 and 102. These read:

$$\int_0^\infty \frac{\ln x}{x^4+1}dx = \frac{-\pi^2\sqrt{2}}{16} = -0 \cdot 87236$$

$$\int_0^\infty \frac{(\ln x)^2}{x^4+1}dx = \frac{3\pi^3\sqrt{2}}{64} = 2 \cdot 055445$$

$$\int_0^\infty \frac{\ln x}{(x^2+1)^2}dx = -\frac{\pi}{4} = -0 \cdot 785398$$

The last answer is mine – Spiegel gets a different answer which does not agree with numerical integration and I am convinced is incorrect.

$$\int_0^\infty \frac{\ln(x+1)}{x^2+1}dx = 1 \cdot 4603619$$

This is similar to the Putnam challenge integral, except with infinity as the upper limit. The numerical value here again is mine – Spiegel's answer of $\frac{1}{2}\pi \ln 2$ is incorrect. I derive this result in full in §5. Finally Spiegel gives (Q.102)

$$\int_0^\pi \ln(r^2 - 2r\cos\theta + 1)d\theta = \begin{cases} 0 & \text{if } |r| \le 1 \\ \pi \ln r^2 & \text{if } |r| \ge 1 \end{cases}$$

From this he deduces that

$$\int_0^{\frac{\pi}{2}} \ln \sin x dx = \int_0^{\frac{\pi}{2}} \ln \cos x dx = -\frac{\pi}{2} \ln 2 \ .$$

4.2 A.S. Holland: 'Complex Function Theory' 1980. On page 191 Holland does in full the interesting case

$$\int_0^\infty \frac{\ln x}{(x+a)^2+b^2}dx = \frac{1}{2b}\ln(a^2+b^2)\tan^{-1}\left(\frac{b}{a}\right) \qquad b > 0. \qquad Eq.4.1$$

The particular case of $a = -1, b = 1$ sheds some light on integrals on the Putnam integrand with different limits. Holland's formula gives

$$\int_0^\infty \frac{\ln x}{x^2-2x+2}dx = \frac{3\pi}{8}\ln 2 = \int_{-1}^\infty \frac{\ln(u+1)}{u^2+1}du.$$

where I have used the substitution $x = u + 1$. This is precisely 3 times the value of the Putnam integral, whose limits are 0 to 1. Recall from §3.1 the numerical value of $1 \cdot 4603619$ when the limits are 0 to ∞.

4.3 Alan Jeffrey: 'Complex Analysis and Applications' 1992. He works in detail the example

$$\int_0^\infty \frac{\ln x}{x^2 + b^2} dx = \frac{1}{2b} \ln b$$

which is Holland's case with $a = 0$, though he uses a different contour.

4.4 E. T. Copson : 'Introduction to Theory of Functions of a Complex Variable' Cambridge Univ. Press, 1935. On page 154 he asks the reader to show that

$$\int_0^\infty \frac{(\ln x)^2}{1 + x^2} dx = \frac{\pi^3}{8}$$

and on page 155 sets an exercise to show the pair

$$\int_0^\infty \frac{\ln x}{(1 + x^2)^2} dx = -\frac{\pi}{4} \quad , \qquad \int_0^\infty \frac{dx}{(1 + x^2)^2} = \frac{\pi}{4} \ .$$

4.5 G. N. Watson : 'Complex Integration and Cauchy's Theorem' Cambridge Univ. Press, 1914. An early textbook by a master of the subject, but not many log integrals are quoted.

4.6 T. M. MacRobert: 'Functions of a Complex Variable' 3rd edition, MacMillan, 1947. A good section on techniques of contour integration with several examples worked in detail and many challenging questions set as exercises, including showing that:

$$\int_0^\infty \frac{\ln x}{x^2 + 1} dx = 0. \qquad Eq.4.2a$$

This is solved as a fully worked example by Spiegel (page 194) and is a special case of the integrals above done by Holland and by Jeffrey, with $a = 0, b = 1$. Note that since $x^2 + 1 > 0$ and $\ln x$ changes sign at $x = 1$, this must mean that

$$-\int_0^1 \frac{\ln x}{x^2 + 1} dx = \int_1^\infty \frac{\ln x}{x^2 + 1} dx = 0 \cdot 915965594 \ldots \qquad Eq.4.2b$$

the numerical value being mine (see footnote). I discuss this integral in §5 below. MacRobert also quotes the result, derived in full by Spiegel, that

$$\int_0^\infty \frac{\ln(x^2 + 1)}{x^2 + 1} dx = \int_0^\infty \frac{\ln(x + 1/x)}{x^2 + 1} dx = \pi \ln 2$$

and the more complicated result

$$\int_0^\infty \frac{(\ln x)^2}{x^2 + x + 1} dx = \frac{16\pi^3}{81\sqrt{3}} \ .$$

I have used four numerical and algebraic software packages to evaluate integrals by quadrature and to check algebra. They are Reduce 3.7, Macsyma, Maxima 5.13 (which is the open source freeware version of Macsyma), and Mathematica 5.

4.7 I. G. Gradshteyn and I. W. Ryzhik : 'Tables of Integrals, Series and Products' Translated by Alan Jeffrey, Academic Press 1965. This classic volume of collected integrals contains loads of further cases. The Putnam problem is given in §4.291.8, page 555. At 4.291.9 they quote the Putnam integrand with limits 0 to ∞ and give the value $\frac{\pi}{4}\ln 2 + \mathbf{G}$ where $\mathbf{G}$ is Catalan's constant, $0\cdot 915965594\ldots$. This agrees with my value of $1 \cdot 4603619$ obtained by numerical integration, as quoted in §4.1. In §5.3 I rise to the challenge of deriving this result in the context of discussing Catalan's constant, which occurs in other integrals involving the logarithm.

In their book, §4.311 and 4.312, Gradshteyn and Ryzhik cite seven intriguing integrals involving $\ln(x^3+1)$. Here are five of them:

$$\int_0^\infty \frac{\ln(a^3 - x^3)}{x^3}\,dx = \frac{4\pi\sqrt{3}}{a^2} \qquad Eq.4.3a$$

$$\int_0^\infty \frac{\ln(x^3+1)}{x^2-x+1}\,dx = \frac{2\pi}{\sqrt{3}}\ln 3 \qquad \int_0^\infty \ln(x^3+1)\frac{1-x}{x^3+1}\,dx = -\frac{2}{9}\pi^2 \qquad Eq.4.3b,c$$

$$\int_0^\infty \frac{\ln(x^3+1)}{x^3+1}\,dx = \frac{\pi}{\sqrt{3}}\ln 3 - \frac{\pi^2}{9} \qquad \int_0^\infty \frac{x\ln(x^3+1)}{x^3+1}\,dx = \pi\sqrt{3}\ln 3 + \frac{\pi^2}{9} \qquad Eq.4.3d,e$$

They give no clue as to how these results were arrived at, and their two references are also obscure. One is to a much older table of integrals compiled by Bierens de Haan and published, in French, in Amsterdam in 1867 (admittedly it was republished in the USA in 1956). This was probably the first attempt to compile all known integrals. But Bierens de Haan does not explain how he obtained these results, nor is there any help from the book of commentary and corrections made on de Haan's tables by C. F. Lindman, published in Sweden in 1891, and reprinted by Stechert, N.Y., 1944. (Lindman is incorrectly referenced as C. E. Lindeman in Gradshteyn and Ryzhik's book.) I find it thought provoking to realise how much effort was put into developing the integral calculus by 19th century mathematicians. However, their techniques are not set out so, in a journey of rediscovery, in §10.4 I rederive the last two of these, and indeed extend to $\int_0^\infty \ln(x^p+1)/(x^p+a^p)\,dx$ for $p=3$ and greater.

4.8 Paul Koosis : 'The Logarithmic Integral' (in two volumes). CUP. 1988. No 12 in Cambridge Studies in Advanced Mathematics. I found this book in the university library and it looked very promising since it discusses integrals of $\ln F(t)/(t^2+1)\,dt$. Its first few lines state "On making the substitution $t = \tan(\theta/2)$ and then putting $F(t) = P(\theta)$, the expression

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\ln F(t)}{t^2+1}\,dt \quad \text{goes over into} \quad \frac{1}{2\pi}\int_{-\pi}^{\pi} \ln P(\theta)\,d\theta. \text{"}$$

This seems like a clever simplification of the Putnam-type integrals. However, I have not found it to be of much practical value since the substitution turns the argument of the logarithm into a complicated trigonometric function with singularities buried inside it. Unfortunately, the first two lines were about as much of the book as I could understand! It is a dense, highly erudite, research level book on mathematical analysis by an expert steeped in the subject. Moreover, as far as I could discern, not one specific integral is evaluated in the whole of volume 1. I must leave this book to the more informed student.

5. Catalan's constant G and related series and integrals

5.1 Catalan's constant and $\int_1^\infty \frac{\ln x}{x^2+1}\,dx$

According to page 807 of the authoritative *'Handbook of Mathematical Functions' by Abramowitz and Stegun, Pub. Dover*, Catalan's constant is defined by the alternating sum of reciprocal odd squares:

$$\mathbf{G} = 1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \ldots = 0 \cdot 91596559417722 \text{ldots}. \qquad Eq.5.1$$

In this section I show that this is the value of $\int_1^\infty \frac{\ln x}{x^2+1} dx$.

For $x > 1$ use the binomial theorem and the substitution $u = 1/x$ to write

$$\frac{1}{x^2+1} = \frac{1}{x^2(1+\frac{1}{x^2})} = \frac{1}{x^2}(1 - \frac{1}{x^2} + \frac{1}{x^4} - \frac{1}{x^6} + \ldots)$$

Hence

$$\int_1^\infty \frac{\ln x}{x^2+1} dx = \int_1^\infty \left[\frac{\ln x}{x^2} - \frac{\ln x}{x^4} + \frac{\ln x}{x^6} - \cdots\right] dx$$

This series is uniformly convergent for $1/x > 1$, this condition being necessary for term by term integration of an infinite series. Even though the binomial expansion is valid strictly only for $1/x > 1$, the above series is valid at the limit $x = 1$ because $\ln 1 = 0$. Fortunately each term can be integrated by parts using $u = \ln x, dv = x^{-n} dx$:

$$\int \frac{\ln x}{x^n} dx = \frac{-1}{(n-1)x^{n-1}} \left[\ln x + \frac{1}{n-1}\right]$$

Hence

$$\int_1^\infty \frac{\ln x}{x^n} dx = \frac{1}{(n-1)^2} \qquad Eq.5.2$$

and Catalan's constant follows immediately. A glance back at Eq. 4.2b will remind the reader that $\int_0^1 \frac{\ln x}{x^2+1} dx = -\mathbf{G}$.

5.2 Some other integrals evaluating to **G**

We can use the above result together with Holland's general formula, Eq. 4.1, to evaluate two other log integrals to augment our collection. In Eq. 4.1 put $a = b = 1$ and make the substitution $x = u - 1$, $(x+1)^2 + 1 = u^2 + 1$. This gives

$$\int_1^\infty \frac{\ln(u-1)}{u^2+1} du = \frac{\pi}{8} \ln 2 = 0 \cdot 2721983.$$

Make the further substitution $u = 1/w, du = -1/w^2.dw$ to obtain

$$\int_1^0 \frac{\ln\left(\frac{1}{w} - 1\right)}{\frac{1}{w^2}+1} \frac{-dw}{w^2} = \int_0^1 \frac{\ln\left(\frac{1}{w} - 1\right)}{1+w^2} dw = \int_0^1 \frac{\ln(1-w) - \ln w}{1+w^2} dw = \int_0^1 \frac{\ln(1-w)}{1+w^2} dw + \mathbf{G},$$

the value **G** coming from Eq. 4.2b. With a change of variable back to x we arrive at

$$\int_0^1 \frac{\ln(1-x)}{x^2+1} dx = \frac{\pi}{8} \ln 2 - \mathbf{G} = -0 \cdot 643767,$$

a result confirmed by numerical integration. This can be added directly to the Putnam integral to obtain, as a bonus,

$$\int_0^1 \frac{\ln(1-x^2)}{x^2+1}dx = \frac{\pi}{4}\ln 2 - \mathbf{G} = -0\cdot 371569.$$

Finally, in trying to determine a quite separate integral by contour integration (which failed to yield the desired result! – see footnote.) I did by accident find that

$$\int_0^1 \frac{\tan^{-1} x}{x}\,dx = \mathbf{G}.$$

5.3 Log integrals and series related to the Riemann Zeta function

Abramowitz and Stegun, page 69, give these two integrals:

$$\int_0^1 \frac{\ln x}{1-x}dx = -\frac{\pi^2}{6} \quad \text{and} \quad \int_0^1 \frac{\ln x}{1+x}dx = -\frac{\pi^2}{12}\ .$$

We can evaluate these by expanding the denominators as binomial series, as was done in §5.1, then integrating term by term between 0 and $1-\varepsilon$, then taking the limit $\varepsilon \to 0$. The relevant binomial expansion is

$$(1 \mp x)^{-1} = 1 \pm x + x^2 \pm x^3 + x^4 \pm x^5 + \cdots$$

Now use $\int_0^1 x^n \ln x\,dx = -1/(n+1)^2$ (obtained by integrating by parts) to find

$$-\int_0^1 \frac{\ln x}{1-x}dx = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \zeta(2) = \frac{\pi^2}{6}$$

$$\text{and} \ -\int_0^1 \frac{\ln x}{1+x}dx = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \eta(2) = \frac{\pi^2}{12}\ .$$

Here $\zeta(x)$ is the Riemann zeta function and $\eta(x)$ the corresponding alternating series, as defined by Abramowitz and Stegun.

Over the range of integration 0 to 1 this method of series expansion can be applied to many integrands of the form $\ln(x)/(\text{polynomial in } x)$. Moreover, the transformation $u = 1/x$ allows related integrals from 1 to ∞ to be obtained. Four examples are

$$-\int_0^1 \frac{\ln x}{x^3+1}dx = \int_1^\infty \frac{x\ln x}{x^3+1}dx = 1 - \frac{1}{4^2} + \frac{1}{7^2} - \frac{1}{10^2} + \cdots = \sum_{n=0}^\infty \frac{(-1)^n}{(3n+1)^2} = 0\cdot 951518$$

$$-\int_0^1 \frac{x\ln x}{x^3+1}dx = \int_1^\infty \frac{\ln x}{x^3+1}dx = \frac{1}{2^2} - \frac{1}{5^2} + \frac{1}{8^2} - \cdots = \sum_{n=0}^\infty \frac{(-1)^n}{(3n+2)^2} = 0\cdot 220436$$

$$\int_0^1 \frac{\ln x}{x^3-1}dx = \int_1^\infty \frac{x\ln x}{x^3-1}dx = \sum_{n=0}^\infty \frac{1}{(3n+1)^2} = 1\cdot 121713.$$

$\oint \frac{\ln(z+1)}{z+1}\,dz$ around a rectangle with corners at $\pm\imath$, $R \pm \imath$ in the limit $R \to \infty$.

$$-\int_0^1 \frac{x^2 \ln x}{x^3+1} dx = \int_1^\infty \frac{\ln x}{x(x^3+1)} dx = \frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{6^2} - \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(3n)^2} = \frac{\pi^2}{108} = 0 \cdot 091385$$

Note in passing that

$$-\int_0^\infty \frac{\ln x}{x^3+1} dx = \sum_{n=0}^{\infty} \left[\frac{(-1)^n}{(3n+1)^2} - \frac{(-1)^n}{(3n+2)^2} \right] = 0 \cdot 951518 - 0 \cdot 220436 = 0 \cdot 7310818 = \frac{2\pi^2}{27}.$$

5.4 Evaluation of $\int_0^\infty \frac{\ln(x+1)}{x^2+1} dx$

Separate this into two integrals over the ranges 0 to 1, and 1 to ∞ respectively:

$$\int_0^\infty \frac{\ln(x+1)}{x^2+1} dx = \int_0^1 \frac{\ln(x+1)}{x^2+1} dx + \int_1^\infty \frac{\{\ln x + \ln(1+\frac{1}{x})\}}{x^2+1} dx.$$

We already have the first integral on the right; it is the Putnam integral, making its reappearance, with value $\frac{\pi}{8} \ln 2$. The first term within the second integral was shown in §5.1 to be Catalan's constant. For the remaining term let $1/x = u$, $dx = -du/u^2$, just as we did above. Then

$$\int_1^\infty \frac{\ln(1+\frac{1}{x})}{x^2+1} dx = \int_1^0 \frac{\ln(1+u)}{\frac{1}{u^2}+1} \left(\frac{-du}{u^2} \right) = \int_0^1 \frac{\ln(1+u)}{u^2+1} du$$

which is the Putnam integral again. Hence

$$\int_0^\infty \frac{\ln(x+1)}{x^2+1} dx = \frac{\pi}{4} \ln 2 + \mathbf{G}. \qquad Eq.5.3$$

5.5 Another example of symmetry in the integrand

We have seen above the effect of the transformation $u = 1/x$ on the integration and we can use it to prove that some integrals are zero. Consider, for example,

$$\int_0^\infty \frac{\ln(x^p+1)}{x^2-qx+1} dx$$

for some positive integer p and real constant q. Make the transformation to u;

$$\int_\infty^0 \frac{\ln(\frac{1}{u^p}+1)}{\frac{1}{u^2}-\frac{q}{u}+1} \frac{-du}{u^2} = \int_0^\infty \frac{\ln(1+u^p) - p\ln u}{u^2-qu+1} du$$

$$\text{so} \quad \int_0^\infty \frac{\ln u}{u^2-qu+1} du \quad \text{must equal } 0$$

for all real values of q. Admittedly (and regrettably!) this tells us nothing about the integral involving $\ln(x^p+1)$, which was merely a device to obtain the above zero value. The result can be obtained from Holland's general result Eq. 3 with $a = -q/2$ and $a^2 + b^2 = 1$.

6. The 'half annulus' and 'key hole' contours for integrating log functions

6.1 A theorem involving $\ln z f(-z)$

I developed this and related theorems from a note in Copson's book, page 154 which reads

> $f(z)$ is a rational function with no poles on the real axis and is such that $zf(z)$ tends to zero as $z \to \infty$ and also as $z \to 0$. By integrating $\ln z f(-z)$ round an appropriate contour, prove that
>
> $$\int_0^\infty \ln z[f(z) + f(-z)]dz + \imath\pi \int_0^\infty f(x)dx \qquad Eq.6.1$$
>
> is equal to $2\pi\imath$ times the sum of residues of $\ln z f(-z)$ at its poles in the upper half plane. The result is of particular value when $f(z)$ is an odd function, as it enables us to evaluate $\int_0^\infty f(x)dx$.

Here is my proof of this, followed in the next sub-section by an example of its application.

Consider evaluating $\oint \ln z f(-z)dz$ around the half-annulus contour in Figure 6.1 below. The indentation at the origin circumvents the singularity in $\ln z$ at zero. The conditions

a) $f(z)$ has no poles on the real axis
b) $zf(z) \to 0$ on γ_2
c) $zf(z) \to 0$ on γ_4

mean that only γ_1 and γ_3 contribute in the limits $\varepsilon \to 0, R \to \infty$. On γ_3 $z = -t$ so

$$\lim_{\varepsilon\to 0, R\to\infty} \int_R^\varepsilon \ln(-t)f(t)d(-t) = \int_0^\infty (\ln t + \imath\pi)f(t)dt.$$

Combine this with $\int_{\gamma_1}$ and apply the Residue Theorem to get Copson's result.

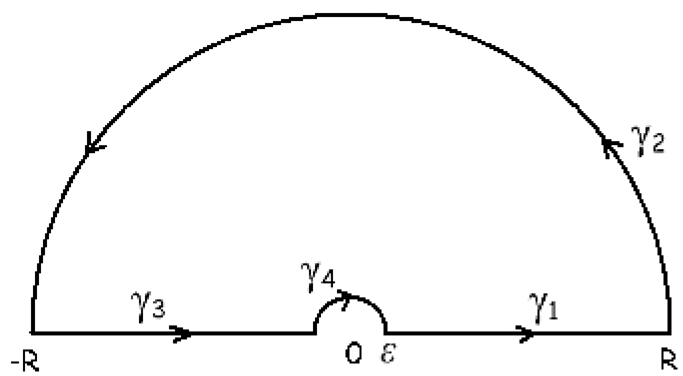

Figure 6.1: Half-annulus contour for evaluating $\oint \ln z f(-z)dz$

For $f(x)$ even there is no advantage in using this formula over straightforward contour integration of $\ln(x)f(x)$. As Copson notes, when $f(x)$ is odd, the real term on the left of Eq. 6.1 is zero and we have $\int_0^\infty f(x)dx$ in terms of a sum of residues. In practice, however, the three conditions on $f(z)$ turn out to be quite restrictive, limiting $f(x)$ mainly to a rational function of polynomials, with the degree of the denominator strictly more than 1 greater than that of the numerator (more on this in §6.5 below). In choosing a worthwhile example to illustrate the technique, I have avoided the many simple rational functions which can be integrated in closed form. I have therefore chosen $f(x) = x/(x^6+1)$ to show the power of the technique, and this is evaluated in the next sub-section.

6.2 An example of Copson's theorem for an odd function: $\displaystyle\int_0^\infty \frac{x}{x^6+1}\,dx$

Let us take $f(x)$ to be the odd function $x/(x^6+1)$. This has no singularity on the real axis, and both limits of $z^2/(z^6+1)$ as $z \to 0$, $z \to \infty$ are zero as required. $f(-z)$ gives rise to six poles of order 1. We want the residues of

$$\frac{(-z)\ln z}{[(-z)^6+1]}$$

within the half-annulus contour, at $z = \imath$, $z = (\pm\sqrt{3}+\imath)/2$. In case the reader is not familiar with evaluating residues, in the Schaum book on *'Theory and Problems of Complex Variables'*, page 172 Murray Spiegel gives the following formula for calculating the residue at a pole of order n positioned at $z = a$:

$$\lim_{z\to a} \frac{1}{(k-1)!}\frac{d^{k-1}}{dz^{k-1}}[(z-a)^k f(z)]$$

$$\text{and with } k = 1 \text{ this reduces to} \qquad \lim_{z\to a}(z-a)f(z).$$

The reader probably knows that this is also obtained (for a pole of order 1) by 'differentiating the denominator':

$$\left.\frac{z}{\frac{d}{dz}(z^6+1)}\right|_{z=a}.$$

The required residue at $\imath$ is

$$\lim_{z\to\imath} \frac{-z\ln z}{z^5+\imath z^4-z^3-\imath z^2+z+\imath} = -(\ln\imath)/6 = -\imath\pi/12.$$

Calculation of the other two residues is tedious but straightforward: at $e^{\imath\pi/6}$ we have $\pi(-\sqrt{3}+\imath)/72$ and at $e^{\imath 5\pi/6}$, $5\pi(\sqrt{3}+\imath)/72$. Adding these we get simply $\pi\sqrt{3}/18$ so Copson's formula immediately gives

$$\int_0^\infty \frac{x}{x^6+1}\,dx = \frac{\pi\sqrt{3}}{9} = 0\cdot 60460,$$

a result confirmed by numerical integration.

6.3 An extension of Copson's theorem to $\oint(\ln z)^2 f(-z)dz$

We again use the half-annulus contour in Figure ??. On γ_2 $z = Re^{\imath\theta}$ and the contribution from this segment is

$$\int_0^\pi \ln^2(Re^{\imath\theta})\,f(-Re^{\imath\theta})\,\imath Re^{\imath\theta}.d\theta$$

which $\to 0$ as $R \to \infty$ provided $zf(z) \to 0$. This is the same condition required for convergence in §6.1. We also require that $\varepsilon f(\varepsilon) \to 0$ on γ_4. Given these conditions only γ_1 and γ_3 contribute.

$$\int_{\gamma_3} = lim_{\varepsilon\to 0, R\to\infty} \int_R^\varepsilon \ln^2(-t)\,f(t)\,(-dt) = \int_0^\infty (\ln t + \imath\pi)^2.f(t).dt$$

Adding the contribution from γ_1 we conclude that

$$\int_0^\infty \ln^2(t)\,[f(t)+f(-t)]dt - \pi^2\int_0^\infty f(t)\,dt + 2\pi\imath\int_0^\infty \ln(t)\,f(t)dt = 2\pi\imath\sum_{residues}\ln^2 z.f(-z) \qquad Eq.6.2$$

Note that for an odd function $f(x)$ this still has non-zero real and imaginary parts which, when equated to the sum over residues, will give *two* integrals, one of which will be $\int_0^\infty f(x)dx$. Thus Eq. 6.2 gives more information at the expense of evaluating more complicated residues. Also note that in Eq. 6.1 we integrate $\ln z\, f(-z)$ around a contour and obtain an integral of $f(x)$ along the real axis, and in Eq. 6.2 we integrate $\ln^2 z\, f(-z)$ around a contour and obtain an integral of $\ln z\, f(x)$ along the real axis. The exponent of $\ln^n z$ is reduced by one. This happens because of cancellation on the paths γ_1 and γ_3. We may suppose that this pattern continues with higher powers of $\ln z$, and indeed in his book (see §4.6) MacRobert obtains $\int_0^\infty \frac{(\ln x)^2}{x^2+x+1}dx$ from $\oint \frac{(\ln z)^3}{z^2+z+1}dz$ around the key hole contour of Figure 6.2 (which we consider in §6.6). This cancellation between contour segments and the consequent reduction in the exponent of $\ln z$ is a distinctive feature of complex integration with the logarithm.

6.4 Example of the $\oint \ln^2 z f(-z)dz$ *formula for an odd function :* $\int_0^\infty \frac{x}{x^6+1}\,dx$

We build on the example in §6.2 by again taking $f(x)$ to be $x/(x^6+1)$. We now want the residues of

$$\frac{-z\ln^2 z}{[(-z)^6+1]}$$

within the half-annulus contour. These can be obtained from the residues in §6.2 by multiplying each by its respective value of $\ln z$ at the pole. We find

at $z=\imath$ residue is $\pi^2/24$,
at $e^{\imath\pi/6}$ residue is $-\pi^2(1+\imath\sqrt{3})/432$.
at $e^{\imath 5\pi/6}$ residue is $25\pi^2(-1+\imath\sqrt{3})/432$.

The sum of these three residues is $\pi^2(-1+\imath 3\sqrt{3})/54$. Now equate real and imaginary parts to obtain immediately from the imaginary part

$$\int_0^\infty \frac{x\ln x}{x^6+1}\,dx = -\frac{\pi^2}{54} = -0\cdot 182770,$$

and, from the real part,

$$\int_0^\infty \frac{x}{x^6+1}\,dx = \frac{\pi^2\sqrt{3}}{9}.$$

I have confirmed the former integral using quadrature with Macsyma, and the latter value is precisely the one obtained in §6.2 using Copson's formula. On balance, the 'log squared' method of this present section seems to yield more results for a given level of effort.

6.5 Integrals involving branch points of multivalued functions: a warning!

The three conditions a), b), c) placed upon $f(z)$ in Copson's theory, §6.1, are met by the function $\sqrt[3]{x}/(x^2+1)$. Moreover, if we think of x as real, this is an odd function; for instance $\sqrt[3]{-8}=-2$. Let us evaluate this in two ways using Eq. 6.2. With the substitution $x=u^3$ it can be transformed into a ratio of polynomials: in fact

$$\int_0^\infty \frac{x^{1/3}}{x^2+1}\,dx = 3\int_0^\infty \frac{u^3}{u^6+1}\,du = 1\cdot 81380,$$

$$\int_0^\infty \frac{x^{1/3}\ln x}{x^2+1}\,dx = 9\int_0^\infty \frac{u^3 \ln u}{u^6+1}\,du = 1\cdot 64455,$$

the values being obtained by numerical quadrature. So these pairs of integrals, over x and u respectively, should be entirely equivalent and we may expect their use within either Eq. 6.1 or Eq. 6.2 to give the same answer. The version in u is similar to the example in §6.4, having the same denominator, and $f(u)$ is certainly odd. I have worked through Eq. 6 with $\oint \frac{(-u)^3 \ln^2 u}{(-u)^6+1}\,du$ and find

$$3\int_0^\infty \frac{u^3}{u^6+1}\,du = \frac{\pi\sqrt{3}}{3}\,, \qquad 9\int_0^\infty \frac{u^3\ln u}{u^6+1}\,du = \frac{\pi^2}{6},$$

in full agreement with the numerical values. Note, in passing, the coincidence that

$$\int_0^\infty \frac{u}{u^6+1}\,du = \int_0^\infty \frac{u^3}{u^6+1}\,du = \frac{\pi\sqrt{3}}{9}\,, \qquad \int_0^\infty \frac{u\ln u}{u^6+1}\,du = -\int_0^\infty \frac{u^3 \ln u}{u^6+1}\,du = -\frac{\pi^2}{54}.$$

But what if we attempt the x version of the integral, which has the multi-valued factor $x^{1/3}$? The denominator of $(-z)^{1/3}\ln^2 z/[(-z)^2+1]$ gives rise to a single pole of order 1 within the half annulus contour, at $z=\imath$. The formula for the residue at this simple (order 1) pole at $z=\imath$, in Spiegel's form quoted in §6.2, is $\lim_{z\to\imath}(z-\imath)f(z)$. This gives a residue of

$$\frac{(-\imath)^{1/3}\ln^2(\imath)}{2\imath} = \imath\pi^2(-\imath)^{1/3}/8,$$

but which of the three roots $(-\imath)^{1/3} = (e^{3\pi\imath/2})^{1/3}$ are we to choose? These roots are $\imath$, $e^{7\pi\imath/6}$ and $e^{11\pi\imath/6}$. As an 'experiment', let us calculate the residue assuming each in turn is correct:

Root $r_1 = \imath$ gives residue $-\pi^2/8$.
Root $r_2 = e^{7\pi\imath/6}$ gives residue $\pi^2[1-\imath\sqrt{3}]/16$.
Root $r_3 = e^{11\pi\imath/6}$ gives residue $\pi^2[1+\imath\sqrt{3}]/16$.

In fact *none* of these gives the correct answer when placed in Eq. 6.2 ! For example r_1, having zero imaginary part, implies that $\int_0^\infty x^{1/3}/(x^2+1)dx = 0$. What can be going on?

The explanation seems to be that $\sqrt[3]{z}/(z^2+1)$ does not behave as a straightforward odd function when taken into the complex plane. For z on the negative real axis, at -8 say, the principal value of the cube root is not -2 but $2e^{\imath\pi/3} = 1+\imath\sqrt{3}$. Consequently the $f(z)+f(-z)$ in the first term of Eq. 6.2 is not zero and the method fails. The method would appear to be restricted to rational functions of polynomials, where the degree of the numerator is at least 2 less than the degree of the denominator.

6.6 Integrals of $\oint \ln z\, f(z)dz$ round a 'key hole' contour

The material in this and the next sub-sections is equivalent to Eqs. 5 and 6 except that the contour is now that in Figure 6.2 below. Because of its shape, it is called a 'key hole' contour. Note that here we are integrating $\ln z\, f(z)$ and not $\ln z\, f(-z)$. As previously in §6 we place the three conditions that a) $f(z)$ has no poles on the real axis, b) $zf(z)\to 0$ as $R\to\infty$ on γ_2, and c) $zf(z)\to 0$ as $\varepsilon\to 0$ on γ_4. Then only γ_1 and γ_3 contribute. Our parameterisation on γ_3 is $z = te^{2\pi\imath}$ so

$$\int_{\gamma_3}: \qquad \lim_{\varepsilon\to 0, R\to\infty}\int_R^\varepsilon \ln(te^{2\pi\imath})\,f(te^{2\pi\imath})\,e^{2\pi\imath}dt = -\int_0^\infty [\ln t + 2\pi\imath]f(t)dt$$

.

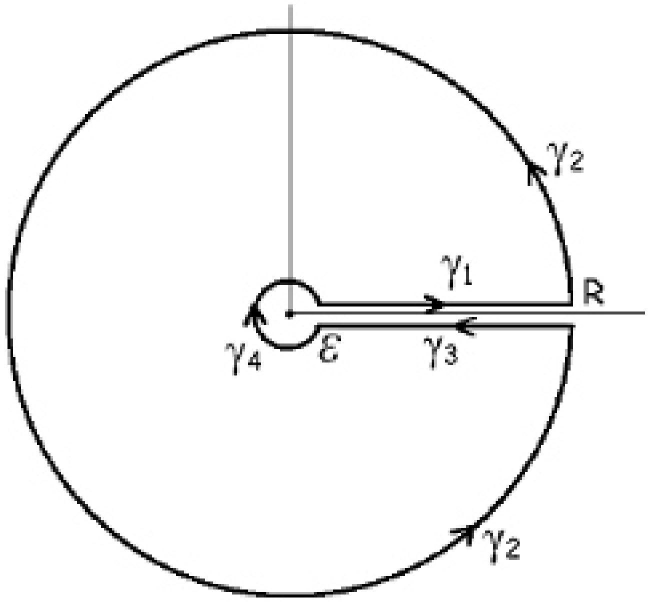

Figure 6.2 : Key hole contour with branch cut for $\ln z$ along the real axis.

and the $\ln t\, f(t)$ cancels with $\int_{\gamma_1}$, leaving

$$-2\pi\imath \int_0^\infty f(t)\,dt = \oint \ln z\, f(z)\,dz.$$

Then

$$\int_0^\infty f(t)\,dt = -\Re \sum_{residues} \ln z\, f(z) \qquad Eq.6.3$$

where $\Re$ denotes the real part, and the sum is over all residues within the key hole contour. As a check on calculations, the imaginary part should be zero.

As an example of using Eq. 6.3, let $f(x) = 1/(x^3+1)$. This has three simple poles at $z = -1, e^{\imath\pi/3}, e^{5\imath\pi/3}$. As a warning to the unwary, note that in this type of work $e^{5\imath\pi/3}$ is *not* the same as $e^{-\imath\pi/3}$ even though they may plot at the same point on the Argand diagram. In the key hole contour the branch cut for $\ln z$ is along the real axis so each pole must be approached solely by anticlockwise travel from the upper side of the real axis at γ_1. The correct evaluation of the residues are respectively $\imath\pi/3$, $\pi(\sqrt{3}-\imath)/18$ and $-5\pi(\sqrt{3}+\imath)/18$, and their sum is $-2\pi\sqrt{3}/9$. There is no imaginary part, as required. Hence from Eq. 6.3

$$\int_0^\infty \frac{dx}{x^3+1}\,dx = \frac{2\pi\sqrt{3}}{9} = 1\cdot 2092,$$

a value checked by numerical quadrature using Macsyma.

6.7 Integrals of $\oint \ln^2 z\, f(z)\,dz$ round the 'key hole' contour

We impose the same conditions a), b), c) on $f(z)$ and integrate $\ln^2 z f(z)$ round the key hole.

$$\int_{\gamma_3} : -\int_0^\infty [\ln t + 2\pi\imath]^2 f(t)\,dt = -\int_0^\infty \ln^2 t\, f(t)dt - 4\imath\pi \int_0^\infty \ln t\, f(t)dt + 4\pi^2 \int_0^\infty f(t)\,dt$$

$$= 2\imath\pi \sum_{residues} \ln^2 z\, f(z)$$

The first term, in $\ln^2$, exactly cancels the contribution from γ_1 and we obtain the quite useful (in this context!) result:

$$\int_0^\infty \ln t\, f(t)\, dt = -\frac{1}{2}\Re \sum_{residues} \ln^2 z\, f(z), \qquad Eq.6.4a$$

$$\int_0^\infty f(t)\, dt = -\frac{1}{2\pi}\Im \sum_{residues} \ln^2 z\, f(z). \qquad Eq.6.4b$$

As an example, let's again take $f(x) = 1/(x^3+1)$. The three residues are:

residue at $z = -1$ is $\pi^2/3$,

residue at $e^{\imath\pi/3}$ is $\pi^2(1+\imath\sqrt{3})/54$,

residue at $e^{5\imath\pi/3}$ is $25\pi^2(1-\imath\sqrt{3})/54$,

with sum $4\pi^2(1-\imath 3\sqrt{3})/27$. We obtain

$$\int_0^\infty \frac{\ln x}{x^3+1}\, dx = -\frac{2\pi^2}{27} = -0\cdot 73108\ , \qquad \int_0^\infty \frac{1}{x^3+1}\, dx = \frac{2\pi\sqrt{3}}{9}$$

the latter result being precisely that obtained using Eq. 6.3. Recall that we obtained this in §5.2. Note here again the phenomenon we saw in §6.1, §6.3 and §6.6 whereby there is cancellation between segments of the contours of the terms involving the highest power of $\ln z$, with consequent reduction of the exponent in $\ln x$ in the final real integral.

6.8 Integrals involving the logarithm and $\dfrac{1}{(x^4+a^4)}$

I have evaluated the following integrals involving the rational function $1/(x^4+a^4)$ and the logarithm:

$$\int_0^\infty \frac{1}{x^4+a^4}\, dx = \frac{\pi\sqrt{2}}{4a^3}$$

$$\int_0^\infty \frac{\ln x}{x^4+a^4}\, dx = \frac{\sqrt{2}}{8a^3}\left(2\pi \ln a - \frac{\pi^2}{2}\right)$$

$$\int_0^\infty \frac{\ln^2 x}{x^4+a^4}\, dx = \frac{\pi\sqrt{2}}{a^3}\left(\frac{3\pi^2}{8} - \pi \ln a + 2\ln^2 a\right)$$

The first was determined by integrating $1/(z^4+a^4)$ around the semi-annulus contour of Figure 3, and the other two by integrating $\ln^2 z/(z^4+a^4)$ around the same contour. This follows the hint given in the Schaum book by Murray Spiegel, page 198, Q 87. Note that by integrating $\ln^2(+z)/(z^4+a^4)$, rather than $\ln^2(-z)/(z^4+a^4)$ as in §6.4, the $\ln^2$ terms do *not* cancel. This approach can be used to determine similar integrals involving $\ln^2 x\, \times$(rational function).

6.9 Integrals of $\dfrac{1}{x^p+a^p}$ *for general integer* $p \geq 2$

Finally I use the key-hole contour to evaluate the general case

$$\int_0^\infty \frac{1}{(x^p+a^p)}\, dx \qquad p \text{ a positive integer} \geq 2.$$

According to Eq. 6.4 this is $-\Re\sum$ (residues of $\frac{\ln z}{z^p+a^p}$ within key-hole contour). The poles are at $z = a\exp[\imath\pi(2k+1)/p]$ for $k = 0, 1, \ldots, p-1$. The residue at $a\exp[\imath\pi(2k+1)/p]$ is

$$\frac{\ln a + \frac{\imath\pi}{p}(2k+1)}{p\,(a\,e^{\imath\pi(2k+1)/p})^{p-1}} = -\frac{e^{-\imath\pi/p}}{p\,a^{p-1}}\,e^{-2\imath\pi/p}\left[\ln a + \frac{\imath\pi}{p}(2k+1)\right]$$

so the sum over all residues is

$$-\frac{e^{-\imath\pi/p}}{p\,a^{p-1}}\sum_{k=0}^{p-1} e^{-\imath 2k\pi/p}\left(\ln a + \imath\frac{\pi}{p}(2k+1)\right) = -\frac{e^{-\imath\pi/p}}{p\,a^{p-1}}\left[\left(\ln a + \imath\frac{\pi}{p}\right)\sum_{k=0}^{p-1} e^{-\imath 2k\pi/p} + \imath\frac{2\pi}{p}\sum_{k=0}^{p-1} k\,e^{-\imath 2k\pi/p}\right].$$

The first sum is a geometric series whose value is zero. The second is an arithmetico-geometric progression whose sum is

$$\sum_{k=0}^{p-1} k\,e^{-\imath 2k\pi/p} = -\frac{p}{1-e^{-\imath 2\pi/p}} = -\frac{p}{2}\left[1 - \imath\frac{\sin(\frac{2\pi}{p})}{1-\cos(\frac{2\pi}{p})}\right].$$

The sum over all the residues is therefore

$$\frac{e^{-\imath\pi/p}}{pa^{p-1}}\,\imath\pi\left[1 - \imath\frac{\sin(\frac{2\pi}{p})}{1-\cos(\frac{2\pi}{p})}\right].$$

In multiplying this out one finds that the imaginary part is zero on account of a trigonometric identity – this is a reassuring sign that the calculation is correct. Finally, the value of the integral is

$$\int_0^\infty \frac{1}{x^p + a^p}\,dx = \frac{\pi}{pa^{p-1}}\left[\frac{2\sin(\frac{\pi}{p})}{1-\cos(\frac{2\pi}{p})}\right] \qquad \text{for } p \text{ a positive integer} \geq 2.$$

I have checked the following numerical values for $a = 1$ against numerical quadrature:

p = 2 :	$\frac{\pi}{2}$	$= 1\cdot 5707963$	p = 3 :	$\frac{2\,\pi\,\sqrt{3}}{9}$	$= 1\cdot 2091996$
p = 4 :	$\frac{\pi\sqrt{2}}{4}$	$= 1\cdot 1107207$	p = 5 :	$-\frac{2\,\pi\,\sin\left(\frac{\pi}{5}\right)}{5\left(\cos\left(\frac{2\,\pi}{5}\right)-1\right)}$	$= 1\cdot 0689593$
p = 6 :	$\frac{\pi}{6}$	$= 1\cdot 0471976$	p = 8 :	$\frac{\sqrt{2}\,\pi\,\sin\left(\frac{\pi}{8}\right)}{4\left(\sqrt{2}-1\right)}$	$= 1\cdot 0261721$

7. The integrals $\int_0^\infty \frac{1}{x^3-1}\,dx$ and $\int_0^\infty \frac{\ln x}{x^3-1}\,dx$

These are an interesting pair of integrals because of the deformations of the contour needed to avoid the singularities. We use the 'trick' in log integration of taking the exponent of $\ln z$ to be one higher than that in the real integral required. Thus we will be integrating $\oint \ln^2 z/(z^3-1)dz$.

As a place to start, let us try to use the key hole contour in Figure 6.2. We first need to identify the singularities of the integrand within the whole complex plane. The infinite singularity in $\ln z$ at $z = 0$ is circumvented by the segment γ_4. The roots of $z^3 - 1 = 0$ are at $z = 1$, $e^{2\pi\imath/3} = (-1+\imath\sqrt{3})/2$, $e^{4\pi\imath/3} = (-1-\imath\sqrt{3})/2$ and the latter two clearly give rise to simple poles inside the key hole contour.

But what of $z = 1$? This is also where $\ln z \to 0$. This point is on the real axis and so lies within the branch cut of $\ln z$, so how will the contour segments γ_1 and γ_3 be affected?

On γ_1: Here $\ln z$ is real. To analyse its behaviour near $z = 1$ write $u = z - 1$ and expand into the well known Taylor series

$$\ln(u + 1) = u - \frac{u^2}{2} + \frac{u^3}{3} - \frac{u^4}{4} + \cdots,$$

valid for $-1 < u \leq 1$. Also write $z^3 - 1 = u^3 - 3u^2 + 3u = 3u(1 + u + u^2/3)$ and expand by the binomial theorem:

$$(z^3 - 1)^{-1} = \frac{1}{3u}\left[1 - \left(u + \frac{u^2}{3}\right) + \left(u + \frac{u^2}{3}\right)^2 - \left(u + \frac{u^2}{3}\right)^3 + \cdots\right]$$

$$\text{so} \qquad \frac{\ln^2 z}{z^3 - 1} = \frac{1}{3u}(1 - u + \cdots)\left(u - \frac{u^2}{3} + \cdots\right)^2 = \frac{u}{3} + au^2 + bu^3 + \cdots$$

where $au^2 + bu^3 + \cdots$ denotes a series in positive powers of u. This function is smooth and continuous, and is zero at $z = 1$. We have shown that this point is a removable singularity, and consequently no change is required to the key hole contour along segment γ_1.

On γ_3: Here z can be parameterised as $te^{2\pi\imath}$, t real, meaning that $\ln^2 z = \ln^2 t - 4\pi^2 + 4\pi\imath \ln t$. The term within $(\ln^2 z)/(z^3 - 1)$ involving $\ln^2 t$ is a removable singularity and so would occasion no change in contour, but the other two terms create a pole of order 1. The contour must therefore be indented at $z = 1$ on the segment γ_3; call this semicircular indent γ_5 and let it have radius ε as in Figure 7.1 below.

Now that the singularities and the contour are established, we can carry out a straightforward evaluation of the residues at the two poles, and the contributions on each segment. The residues are found from the 'differentiate the denominator' rule, by which we evaluate $\ln^2 z/(3z^2)$ at each pole in turn. At $e^{2\pi\imath/3}$ the residue is $2\pi^2(1 - \imath\sqrt{3})/27$ and at $e^{4\pi\imath/3}$ the residue is $8\pi^2(1 + \imath\sqrt{3})/27$, giving a sum of $2\pi^2(5 + \imath 3\sqrt{3})$ and a value of the loop integral of $4\pi^3(-3\sqrt{3} + 5\imath)/27$, from the residue theorem.

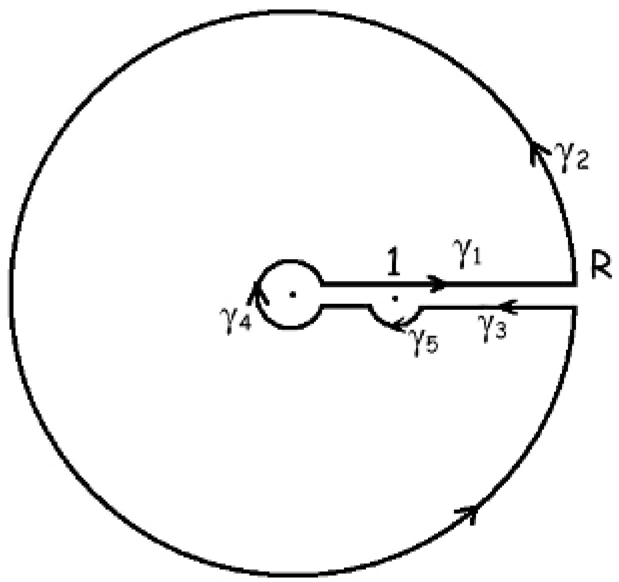

Figure 7.1 : Key hole contour modified at $z = 1$ for $\oint \frac{\ln^2 z}{z^3 - 1} dz$

The contributions from the segments of the contour are :

On γ_1 : $z = t$ and in the limits $\varepsilon \to 0$, $R \to \infty$ $\int_{\gamma_1} = \int_0^\infty \ln^2 t/(t^3-1)dt$

On γ_2 : In the limit $R \to \infty$ this contribution is zero.

On γ_3 : As described above, $z = te^{2\pi\imath}$. In the limits $\varepsilon \to 0$, $R \to \infty$

$$\int_{\gamma_3} = \int_{R,t\neq 1}^{\varepsilon} \frac{\ln(te^{2\pi\imath})^2}{t^3-1}\,dt \to \quad -\int_{0,t\neq 1}^{\infty} \frac{(\ln^2 t - 4\pi^2 + 4\pi\imath)}{t^3-1}\,dt$$

The first term in this integral will exactly cancel the contribution from γ_1.

On γ_4 : $z = \varepsilon e^{\imath\theta}$ so

$$\int_{\gamma_4} = \lim_{\varepsilon\to 0}\int_{2\pi}^{0} \frac{\ln^2(\varepsilon e^{\imath\theta})}{\varepsilon^3 e^{3\imath\theta}-1}\,\imath\varepsilon e^{\imath\theta}\,d\theta = \lim_{\varepsilon\to 0}\int_0^{2\pi} (\ln\varepsilon + \imath\theta)^2\,\imath\varepsilon e^{\imath\theta}\,d\theta = 0$$

This result depends on the limit $\varepsilon\ln\varepsilon \to 0$ as $\varepsilon \to 0$, bearing in mind that $\ln\varepsilon$ itself $\to -\infty$. This is equivalent to $\varepsilon^\varepsilon \to 1$ as $\varepsilon \to 0$.

On γ_5 : $z = e^{2\pi\imath} + \varepsilon e^{\imath\theta}$ so

$$\int_{\gamma_5} = \lim_{\varepsilon\to 0}\int_{2\pi}^{\pi} \frac{\ln^2(e^{2\pi\imath}+\varepsilon e^{\imath\theta})}{3\varepsilon e^{\imath\theta}+3\varepsilon^2 e^{2\imath\theta}+\varepsilon^3 e^{3\imath\theta}}\,\imath\varepsilon e^{\imath\theta}\,d\theta = -\lim_{\varepsilon\to 0}\int_{\pi}^{2\pi} \frac{-4\pi^2}{3\varepsilon e^{\imath\theta}}\,\imath\varepsilon e^{\imath\theta}\,d\theta = \frac{4\pi^3\imath}{3}\ .$$

Adding these contributions, equating them to the loop integral, and taking real and imaginary parts gives our required result

$$\int_0^\infty \frac{1}{x^3-1}\,dx = -\frac{\pi\sqrt{3}}{9} = -0\cdot 60460\ , \qquad \int_0^\infty \frac{\ln x}{x^3-1}\,dx = \frac{4\pi^2}{27} = 1\cdot 46216.$$

I have checked both values by numerical quadrature. (Recall that we obtained the equivalent integral from 0 to 1 in §5.2.) In summary, the interest in this semi-infinite integral is the removable singularity on one branch of the real axis but not the other, and the demonstration of how integrating $\oint \ln^2 zf(z)dz$ round the contour yields $\int_0^\infty \ln zf(z)dz$ owing to cancellation of terms in $\ln^2 zf(z)$.

In closing this section, note that the indefinite integral of $1/(x^3-1)$ can be obtained in closed form as follows. First split it into partial fractions:

$$\frac{1}{x^3-1} = \frac{1}{3(x-1)} - \frac{2x+1}{6(x^2+x+1)} - \frac{1}{2(x^2+x+1)}.$$

Next complete the square on $x^2+x+1 = (x+\frac{1}{2})^2 + \frac{3}{4}$, then integrate to obtain

$$\int \frac{dx}{x^3-1} = \tfrac{1}{3}\ln(x-1) - \tfrac{1}{6}\ln(x^2+x+1) - \tfrac{1}{\sqrt{3}}\tan^{-1}\left(\tfrac{2x+1}{\sqrt{3}}\right) + \text{arbitrary constant.}$$

However this breaks down at $x = 1$ and so cannot be used to evaluate the integral $\int_0^\infty$.

8. A table of integrals generalising the Putnam integral

This section and the next deal with a generalisation of the Putnam integral to

$$\int_0^\infty \frac{\ln(x^p+1)}{x^2+1}dx \qquad Eq.8.1$$

where p is either a positive integer or a fraction. In this section I give a table of values of these integrals, and in look at factorisation of x^p+1 as one possible approach to breaking down this complicated integral into a sum of simpler, more tractable integrals. In §9 I attempt to evaluate some of these integrals for various values of p ; the challenge is to make p as general as possible.

8.1 Table of values of $\int_0^\infty \frac{\ln(x^p+1)}{x^2+1}dx$

Using Macsyma and Reduce, two powerful numerical and symbolic maths packages, I have numerically evaluated integrals of the type of Eq. 8.1 for increasing values of the real parameter p. First, note the form of the integrand, illustrated in Figure 8.1. As p increases the contribution from $0 < x < 1$ decreases towards zero, and the contribution from $x > 1$ steadily increases, with the peak moving closer to $x = 1$ from above.

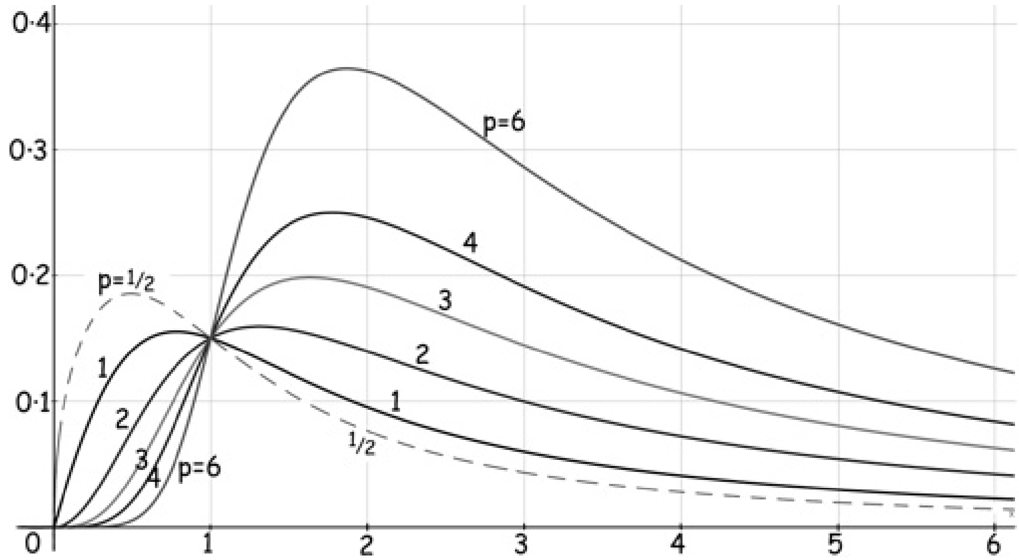

Figure 8.1 : Graphs of $\frac{\ln(x^p+1)}{x^2+1}$ for various values of p

Where I have been able to identify analytical, closed form values of the integral I, I have listed these too in the table over page. **G** is Catalan's constant, $0 \cdot 9159656$. The difference of the last two values in the table is $0 \cdot 916$. A graph of the integral I against p is given in Figure 8.2.

I have calculated the high values of p in Table 8.1 to show that $I(p)$ appears to be asymptotic to a straight line. The line through the points at $p = 40$ and 41 has equation $I = 0.916p + 0.019$ and we may suspect that, for very large values of p the asymptote would be $I = \mathbf{G}p$, where **G** is Catalan's constant. To see how this arises, differentiate the integrand with respect to p :

$$\frac{dI}{dp} = \frac{x^p \ln x}{(x^p+1)(x^2+1)}.$$

Table 8.1

Numerical and algebraic values of generalised Putnam integral between limits 0 to ∞

p	$\int_0^\infty \frac{\ln(x^p+1)}{x^2+1}dx$	formula
0	1·088793	$\frac{\pi}{2}\ln 2 = \pi \ln \sqrt{2}$
$\frac{1}{2}$	1·198674	
1	1·460362	$\frac{\pi}{4}\ln 2 + \mathbf{G}$
$\frac{3}{2}$	1·798631	
2	2·177586	$\pi \ln 2$
$\frac{5}{2}$	2·580109	
3	2·997304	
$\frac{7}{2}$	3·424170	
4	3·857710	$\pi \ln(2+\sqrt{2})$
$\frac{9}{2}$	4·296026	
5	4·737871	
$\frac{11}{2}$	5·182389	
6	5·628978	$\pi \ln 6$
$\frac{13}{2}$	6·077200	
7	6·526732	
8	7·428812	$\pi \ln(4+\sqrt{2}+2\sqrt{2+\sqrt{2}}+2\sqrt{2-\sqrt{2}})$
9	8·333848	
10	9·241000	$\pi \ln(10+4\sqrt{5})$
12	11·059596	$\pi \ln\ (4\sqrt{6}+4\sqrt{3}+5\sqrt{2}+10\ )$
14	12·881929	see Eq. 18
16	14·706627	see Eq. 14
20	18·360314	see Eq. 16
30	27·5063	
32	29·336572	see Eq. 15
40	36·659	see Eq. 16
41	37·575	

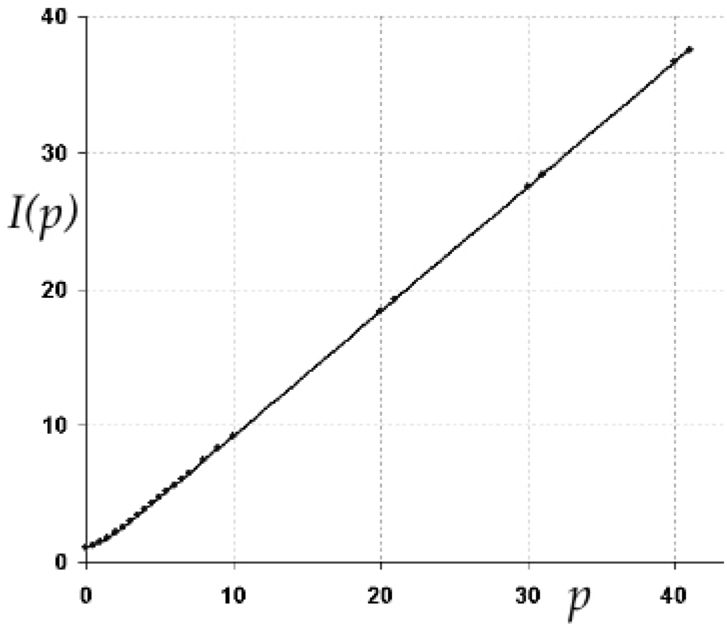

Figure 8.2: Numerical values of the integral $I(p)$ as a function of p

Now take the limit $p \to \infty$ for the two cases $x < 1$ and $x > 1$. For $x < 1, I(p) \to 0$ as $x^p \to 0$. This can be seen from the trend in Figure 8.1 which illustrates how below $x = 1$ the integrand makes negligible contribution for large p. For $x > 1$ the limit is $(\ln x)/(x^2 + 1)$. Consequently the limiting value of dI/dp is given by Eq. 4.2b, with value $\mathbf{G}$.

Just a note about the cases where p is a fraction, $p = k/2$, say, where k an odd integer. The substitution $u^2 = x$ makes

$$\int_0^\infty \frac{\ln(x^{k/2} + 1)}{x^2 + 1} \, dx = 2 \int_0^\infty \frac{u \ln(u^k + 1)}{u^4 + 1} \, du \ .$$

I confess now that, throughout this investigation, I have made no progress with these, and I suspect that they do not evaluate in closed form.

8.2 Table of factors of $x^p + 1$

One way to approach integrals involving $\ln(x^p + 1)$ for integer p is to factorise $x^p + 1$ and use the additive property of logarithms to split the original integral into a sum of simpler integrals which, hopefully, can be evaluated. Factorisation can always be achieved over the complex numbers. For complex z, $z^p + 1$ has p roots equally spaced around the unit circle in the complex plane. Motivated by this fact, in §9 I evaluate $\oint \ln(z + e^{\imath\pi/k})/(z^2 + 1)dz$ around a suitable contour and show how the results for various values of k allow us to build up some of the real integrals of the type in Eq. 9.

The factorisation of $x^p + 1$ over the reals is interesting in its own right and so is demonstrated in Table 8.2 below. These factorisations follow various patterns depending on how p itself factorises, and I have used this property to separate the table into various categories.

1) If $p = 2^n$ for some integer n, x^p is positive and so, by the factor-remainder theorem, $x^p + 1$ is irreducible over the integers, and hence over the rationals.

2) $x^{2^n} + 1$ is a factor whenever $p = 4\times$integer. We see this for $p =$ 12, 20, 24, 36, 48.

3) If $p \neq 2^n$ and p is odd, by the Factor theorem $x + 1$ must be a factor.

4) Similarly, if $p \neq 2^n$ and p is even, $x^2 + 1$ is a factor. In general if p has a prime factor $\mathbf{P}$, the polynomial has a factor $x^{\mathbf{P}} + 1$.

5) For p an odd prime, the pattern is always $(x + 1)(x^{p-1} - x^{p-2} + \cdots + x^2 - x + 1)$,

6) Where p is twice an odd prime the pattern is $(x^2 + 1)(x^{2(p-1)} - x^{2(p-2)} + \cdots + x^4 - x^2 + 1)$.

7) Where p is composite, a frequent factor is the descending sequence $x^{2j} - x^j + 1$ for some integer j.

Table 8.2 : Factorisations of $x^p + 1$ for p an integer

Irreducible polynomials for $p = 2^n$

$$x^2 + 1 \ , \ x^4 + 1 \ , \ x^8 + 1 \ , \ x^{16} + 1 \ , \ x^{32} + 1 \ , \text{ etc..}$$

p and odd prime : $(x + 1)$ is factor

$$(x^3 + 1) \ = \ (x^2 - x + 1)(x + 1)$$

$$\begin{aligned}
(x^5 + 1) &= (x^4 - x^3 + x^2 - x + 1)(x + 1) \\
(x^7 + 1) &= (x^6 - x^5 + x^4 - x^3 + x^2 - x + 1)(x + 1) \\
(x^{11} + 1) &= (x^{10} - x^9 + x^8 \cdots + x^2 - x + 1)(x + 1) \\
(x^{13} + 1) &= (x^{12} - x^{11} + x^{10} \cdots + x^2 - x + 1)(x + 1)
\end{aligned}$$

$$
\begin{aligned}
(x^{17}+1) &= (x^{16}-x^{15}+x^{14}\cdots+x^2-x+1)(x+1)\\
(x^{19}+1) &= (x^{18}-x^{17}+x^{16}\cdots+x^2-x+1)(x+1)\\
(x^{23}+1) &= (x^{22}-x^{21}+x^{20}\cdots+x^2-x+1)(x+1)\\
(x^{29}+1) &= (x^{28}-x^{27}+x^{26}\cdots+x^2-x+1)(x+1)
\end{aligned}
$$

$$\cdots$$

$p=2\times$*(odd prime) :* (x^2+1) *is factor*

$$(x^6+1) = (x^4-x^2+1)(x^2+1)$$

$$
\begin{aligned}
(x^{10}+1) &= (x^8-x^6+x^4-x^2+1)(x^2+1)\\
(x^{14}+1) &= (x^{12}-x^{10}+x^8-x^6+x^4-x^2+1)(x^2+1)\\
(x^{22}+1) &= (x^{20}-x^{18}+x^{16}\cdots+x^4-x^2+1)(x^2+1)\\
(x^{26}+1) &= (x^{24}-x^{22}+x^{20}\cdots+x^4-x^2+1)(x^2+1)
\end{aligned}
$$

$$\cdots$$

$p=3\times$*(odd prime) :* (x^3+1) *is factor*

$$(x^9+1) = (x^6-x^3+1)(x^2-x+1)(x+1)$$

$$(x^{15}+1) = (x^8+x^7-x^5-x^4-x^3+x+1)(x^4-x^3+x^2-x+1)(x^2-x+1)(x+1)$$

$$(x^{21}+1) = (x^{12}+x^{11}-x^9-x^8+x^6-x^4-x^3+x+1)(x^6-x^5+x^4-x^3+x^2-x+1)(x^2-x+1)(x+1)$$

$$(x^{27}+1) = (x^{18}-x^9+1)(x^6-x^3+1)(x^2-x+1)(x+1)$$

$$
\begin{aligned}
(x^{33}+1) = {}&(x^{20}+x^{19}-x^{17}-x^{16}+x^{14}+x^{13}-x^{11}-x^{10}-x^9+x^7\\
&+x^6-x^4-x^3+x+1)(x^{10}-x^9+x^8-\cdots+x^2-x+1)(x^2-x+1)(x+1)
\end{aligned}
$$

$$\cdots$$

$p=5\times$*(odd prime):* (x^5+1) *is factor, with descending alternating sequence*

$$(x^{25}+1) = (x^{20}-x^{15}+x^{10}-x^5+1)(x^4-x^3+x^2-x+1)(x+1)$$

$$
\begin{aligned}
(x^{35}+1) = {}&(x^{24}+x^{23}-x^{19}-x^{18}-x^{17}-x^{16}+x^{14}+x^{13}+x^{12}+x^{11}+x^{10}-x^8-x^7-x^6\\
&-x^5+x+1)(x^6-x^5+x^4-x^3+x^2-x+1)(x^4-x^3+x^2-x+1)(x+1)
\end{aligned}
$$

$$\cdots$$

$p=4m:(x^{2^n}+1)$ *is factor, with alternating descending sequence* $x^{2j}-x^j+1$

$$(x^{12}+1) = (x^8-x^4+1)(x^4+1)$$

$$
\begin{aligned}
(x^{20}+1) &= (x^{16}-x^8+1)(x^4+1)\\
(x^{24}+1) &= (x^{16}-x^8+1)(x^8+1)\\
(x^{48}+1) &= (x^{32}-x^{16}+1)(x^{16}+1)
\end{aligned}
$$

p composite: a few examples, showing alternating descending sequences

$$(x^{18}+1) \;=\; (x^{12}-x^6+1)(x^4-x^2+1)(x^2+1)$$

$$(x^{28}+1) \;=\; (x^{24}-x^{20}\cdots-x^4+1)(x^4+1)$$

$$\begin{aligned}(x^{30}+1) &= (x^{16}+x^{14}-x^{10}-x^8-x^6+x^2+1)\times\\ &\qquad (x^8-x^6+x^4-x^2+1)(x^4-x^2+1)(x^2+1)\end{aligned}$$

$$(x^{36}+1) \;=\; (x^{24}-x^{12}+1)(x^8-x^4+1)(x^4+1)$$

$$\begin{aligned}(x^{42}+1) &= (x^{24}+x^{22}-x^{18}-x^{16}+x^{12}-x^8-x^6+x^2+1)\times\\ &\qquad (x^{12}-x^{10}+x^8-x^6+x^4-x^2+1)(x^4-x^2+1)(x^2+1)\end{aligned}$$

In the next section I show a method for evaluating some integrals involving the factors in the above factorisations, and hence determining $\int_0^\infty \frac{\ln(x^p+1)}{x^2+1}dx$ for several values of integer p.

9. Contour integrals of the form $\oint \dfrac{\ln(z+e^{\imath\pi/k})}{z^2+1}dz$

9.1 Contour integration of $\oint \frac{\ln(z+e^{\imath\pi/k})}{z^2+1}dz$

Prompted by the factorisation of z^p+1 over the complex numbers z, I propose in this section to explore contour integrals of the form

$$\oint \frac{\ln(z+e^{\imath\pi/k})}{z^2+1}dz$$

where k is either an integer or a simple fraction. We hope that suitable choices of k and the contour might lead to values for the real integrals Eq. 8.1, tabulated in §8.2. In fact I will show how this choice leads to closed form expressions for real integrals of the form

$$\int_0^\infty \frac{\ln(t^{4n}+1)}{t^2+1}\,dt$$

and for a class of log(descending polynomial) forms where the degree of the polynomial is either 2 or a multiple of 4. In the following sections, §10 to 12, I attempt with limited success to generalise to the logarithm of other irreducible polynomials.

Let's start by looking at the singular points in the integrand. Clearly the denominator is zero at $z=\pm\imath$ and gives rise to two simple poles. The logarithm tends to $-\infty$ as $z\to -e^{\imath\pi/k}$, and the branch cut must run from this point to infinity in some direction. Unless $k=1$, $\frac{1}{2}$ or some other value which places z on the real axis, the singularity in the logarithm and the branch cut can be kept strictly in the lower half plane. Therefore we can use the simple, non-indented half circle contour in Figure 9.1 There are two ways in which we can regard the segment along the real axis. First, since we are interested in integrals from 0 to $+\infty$, we shall consider it as made of two segments: γ_1 from 0 to $+\infty$ and γ_3 from $-\infty$ to 0. Later, in §9.6, I consider it as one segment parameterised by a single real variable, and so obtain related integrals over the domain $-\infty$ to $+\infty$.

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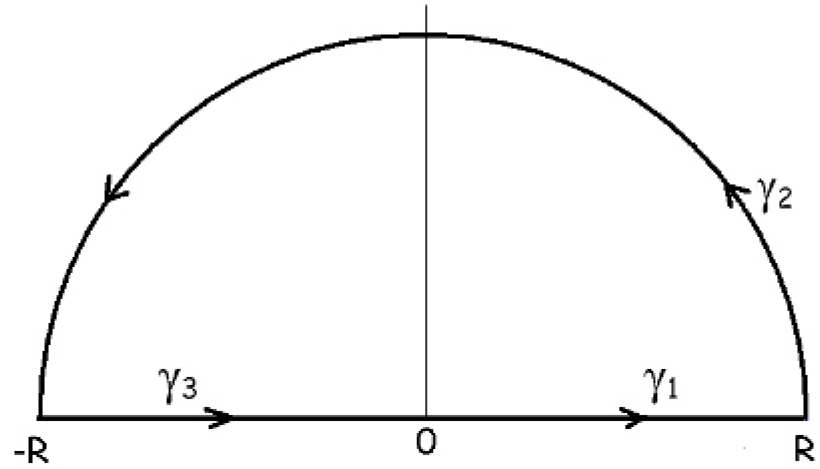

Figure 9.1: Contour used to evaluate $\oint \frac{\ln(z+e^{\imath\pi/k})}{z^2+1}dz$

On γ_1 parameterise z as t, t real. Then

$$\ln|t+e^{\imath\pi/k}|^2 = \ln[(t+e^{\imath\pi/k})(t+e^{-\imath\pi/k})] = \ln[t^2+1+2t\cos\tfrac{\pi}{k}],$$

so the contribution to the loop integral from γ_1 is

$$\frac{1}{2}\int_0^\infty \frac{\ln(t^2+1+2t\cos\frac{\pi}{k})}{t^2+1}dt + \imath\int_0^\infty \frac{\tan^{-1}\left(\frac{\sin\frac{\pi}{k}}{t+\cos\frac{\pi}{k}}\right)}{t^2+1}dt \qquad Eq.9.1a$$

On γ_3 $z=-t$ and the contribution is

$$\frac{1}{2}\int_0^\infty \frac{\ln(t^2+1-2t\cos\frac{\pi}{k})}{t^2+1}dt + \imath\int_0^\infty \frac{\tan^{-1}\left(\frac{\sin\frac{\pi}{k}}{-t+\cos\frac{\pi}{k}}\right)}{t^2+1}dt \qquad Eq.9.1b$$

The contribution from γ_2 is zero in the limit $R\to\infty$. Now note that when we add the contributions from γ_1 and γ_3, their real parts combine into

$$\int_0^\infty \frac{\ln[(t^2+1)^2-4t^2\cos^2\frac{\pi}{k}]}{t^2+1}dt.$$

Only the pole at $z=\imath$ contributes to the loop integral so its value, from the residue theorem, is

$$\oint: \quad 2\pi\imath\frac{\ln(\imath+e^{\imath\pi/k})}{2\imath} = \pi\ln[2+2\sin\tfrac{\pi}{k}] + \imath\pi\tan^{-1}\left(\frac{1+\sin\frac{\pi}{k}}{\cos\frac{\pi}{k}}\right).$$

I concentrate on the real parts since I am interested mainly in integrals involving logarithms, rather than the arctangent ones.

$$\int_0^\infty \frac{\ln[(t^2+1)^2-4t^2\cos^2\frac{\pi}{k}]}{t^2+1}dt = \pi\ln[2+2\sin\tfrac{\pi}{k}] \qquad Eq.9.2$$

This is the key equation for the remainder of the analysis in this section. It is important (though only in this context!) to note that the argument of the logarithm is a fourth degree polynomial; without modification the method cannot give integrals over odd powered polynomials, which would include all odd primes. It will be convenient to introduce the shorthand notation $(t^2+1)^2-4t^2\cos^2\frac{\pi}{k}=F(k)$.

9.2 Some evaluations of $\oint \dfrac{\ln(z+e^{\imath\pi/k})}{z^2+1}\,dz$

Here are some evaluations of Eq. 9.2 for various values of k. These early results hint at patterns which I will explore more systematically in §8.3 and 8.4. All numerical values have been checked by numerical quadrature using Macsyma and Reduce.

Case $k=2$

Since $\cos\frac{\pi}{2}=0$ and $\sin\frac{\pi}{2}=1$, and $\ln(t^2+1)^2 = 2\ln(t^2+1)$ we obtain

$$\int_0^\infty \frac{\ln(t^2+1)}{t^2+1}\,dt = \pi\ln 2.$$

This integral is derived in full as a worked example by Murray Spiegel, §3.1. It is one of the frequent factors identified in §7.2 and we will use it in §8.2 as a building block in forming integrals involving $\ln(x^{2n}+1)$. Note how this degree 2 polynomial has been obtained from the degree 4 polynomial in Eq. 9.2 by virtue of the additive property of the logarithm.

Case $k=3$

Here $\cos\frac{\pi}{3}=\frac{1}{2}$ and $\sin\frac{\pi}{3}=\frac{\sqrt{3}}{2}$. We arrive at

$$\int_0^\infty \frac{\ln(t^4+t^2+1)}{t^2+1}\,dt = \pi\ln(2+\sqrt{3}) = 4\cdot 137345.$$

Precisely the same value is obtained if k is set to $3/2$.

Case $k=4$

Here $\cos\frac{\pi}{4}=\sin\frac{\pi}{4}=\frac{1}{\sqrt{2}}$. We find

$$\int_0^\infty \frac{\ln(t^4+1)}{t^2+1}\,dt = \pi\ln(2+\sqrt{2}).$$

Case $k=6$

Here $\cos\frac{\pi}{6}=\frac{\sqrt{3}}{2}$ and $\sin\frac{\pi}{6}=\frac{1}{2}$, the reverse of $k=3$. We find

$$\int_0^\infty \frac{\ln(t^4-t^2+1)}{t^2+1}\,dt = \pi\ln 3 = 3\cdot 451392.$$

The argument of the logarithm is one of the descending alternating sequences identified as a frequent factor of x^p+1 in §8.2. We will use it as a building block in §9.2. We can use the additive property of the logarithm to combine the integrals for $k=3$ and $k=6$ into

$$\int_0^\infty \frac{\ln(t^8+t^4+1)}{t^2+1}\,dt = \pi\ln[3(2+\sqrt{3})] = 7\cdot 588746. \qquad Eq.9.3$$

Cases $k=8$ *and* $k=\frac{8}{3}$

I am taking these two cases together because I wish to combine their integrals using the additive property of logs. For $k=8$, $\cos^2\frac{\pi}{8}=\frac{2+\sqrt{2}}{4}$ and for $k=\frac{8}{3}$, $cos^2\frac{3\pi}{8}=\frac{2-\sqrt{2}}{4}$. The case $k=8$ makes the argument of the logarithm equal to $t^4+1-\sqrt{2}\,t^2$ whilst $k=\frac{8}{3}$ makes the argument to be $t^4+1+\sqrt{2}\,t^2$. We thus have another 'difference of two squares' with the combined argument $(t^4+1)^2 - 2t^4 = t^8+1$, giving another integral of the form considered in §7. The value of the integral can be expressed in terms of trigonometric functions or in terms of surds (sometimes called 'algebraic irrationalities'):

$$\int_0^\infty \frac{\ln(t^8+1)}{t^2+1}\,dt = \pi \ln[\ 4(1+\sin\tfrac{\pi}{8})\ (1+\sin\tfrac{3\pi}{8})\ ]$$

$$= \pi\ln(\ 4+\sqrt{2}\ +2\sqrt{2+\sqrt{2}}+2\sqrt{2-\sqrt{2}}\ ) = 7\cdot 428812. \qquad Eq.9.4$$

Cases $k=5$ *and* $k=\frac{5}{3}$

This pair of values can be used in a combination following the method above applied to the pair $k=8$ and $k=\frac{8}{3}$. $\cos\frac{\pi}{5} = \frac{1+\sqrt{5}}{4}$ and $\cos\frac{3\pi}{5} = \frac{1-\sqrt{5}}{4}$. Together these give the argument of the logarithm to be the neat sequence $t^8+t^6+t^4+t^2+1$. The sines of $\pi/5$ and $3\pi/5$ can be expressed as surds, but only in the ungainly forms

$$\sin\tfrac{\pi}{5} = \cos\tfrac{3\pi}{10} = \tfrac{1}{2}\sqrt{\tfrac{1}{2}(5-\sqrt{5}} \quad \text{and} \quad \sin\tfrac{3\pi}{5} = \cos\tfrac{\pi}{10} = \tfrac{1}{2}\sqrt{\tfrac{1}{2}(5+\sqrt{5}}.$$

The result is

$$\int_0^\infty \frac{\ln(t^8+t^6+t^4+t^2+1)}{t^2+1}\,dt = \pi\ln\ \Big[\ 4+\sqrt{5}+\sqrt{2}\ (\sqrt{5+\sqrt{5}}\ +\sqrt{5-\sqrt{5}}\ )\ \Big] = 7\cdot 907406.$$

Cases $k=10$ *and* $k=\frac{10}{3}$

We use the facts that $\cos\frac{\pi}{5} = \sin\frac{3\pi}{10} = (1+\sqrt{5})/4$ and $\cos\frac{3\pi}{10} = \sin\frac{\pi}{10} = (1-\sqrt{5})/4$. The calculation is an exact parallel of the pair $k=5$, $k=\frac{5}{3}$ and we obtain the complementary result

$$\int_0^\infty \frac{\ln(t^8-t^6+t^4-t^2+1)}{t^2+1}\,dt = \pi\ln(5+2\sqrt{5}\,) = 7\cdot 063414.$$

This is another alternating descending sequence, so can be used as a building block following the factorisations in §8.2.

Cases $k=12$ *and* $k=\frac{12}{5}$

$\cos\frac{\pi}{12} = \frac{1+\sqrt{3}}{2\sqrt{2}} = \sin\frac{5\pi}{12}$ and $\cos\frac{5\pi}{12} = \frac{-1+\sqrt{3}}{2\sqrt{2}} = \sin\frac{\pi}{12}$. Combining this pair in the same manner as above for $k=8$ and $k=\frac{8}{3}$ we obtain t^8-t^4+1 as the argument of the logarithm. This is another alternating descending sequence. From it

$$\int_0^\infty \frac{\ln(t^8-t^4+1)}{t^2+1}\,dt = \pi\ln(5+2\sqrt{6}) = 7\cdot 201886.$$

This value is subtly similar to that in cases $k=10$, $k=10/3$ above so I am tempted to combine it with Eq. 9.3 from the case $k=6$ and obtain the true but probably totally useless result

$$\int_0^\infty \frac{\ln(t^{16}+t^8+1)}{t^2+1}\,dt = \pi\ln\Big[3(4\sqrt{6}+5\sqrt{3}+6\sqrt{2}+10)\Big] = 14\cdot 790624.$$

On a less frivolous note, if we include with $k = 12$ and $k = \frac{12}{5}$ the intermediate value of $k = \frac{12}{3}$ we obtain the factorisation

$$\begin{aligned} t^{12} + 1 &= [(t^2+1)^2 - 4t^2 \cos^2 \tfrac{\pi}{12}][(t^2+1)^2 - 4t^2 \cos^2 \tfrac{3\pi}{12}][(t^2+1)^2 - 4t^2 \cos^2 \tfrac{5\pi}{12}] \\ &= (t^4 - \sqrt{3}t^2 + 1)(t^4 + 1)(t^4 + \sqrt{3}t^2 + 1) \end{aligned}$$

from which

$$\begin{aligned} \int_0^\infty \frac{\ln(t^{12}+1)}{t^2+1}\, dt &= \pi \ln[\ 2^3 (1 + \sin \tfrac{\pi}{12})\ (1 + \sin \tfrac{3\pi}{12})\ (1 + \sin \tfrac{5\pi}{12})\ ] \qquad Eq.9.5 \\ &= \pi \ln(\ 4\sqrt{6} + 4\sqrt{3} + 5\sqrt{2} + 10\ ) = 11 \cdot 0595963. \end{aligned}$$

Cases $k = 16$ *and related values*

By now a pattern is emerging for integrals involving $t^{4n} + 1$ which derives from a particular factorisation of this function. Eqs. 9.4 and 9.5 gave this factorisation for $n = 2$ and $n = 3$ respectively, and the equivalent factorisation for $n = 4$ is

$$t^{16} + 1 = [(t^2+1)^2 - 4t^2 \cos^2 \tfrac{\pi}{16}][(t^2+1)^2 - 4t^2 \cos^2 \tfrac{3\pi}{16}][(t^2+1)^2 - 4t^2 \cos^2 \tfrac{5\pi}{16}][(t^2+1)^2 - 4t^2 \cos^2 \tfrac{7\pi}{16}]$$

Substituting this into Eq. 10.2 gives

$$\int_0^\infty \frac{\ln(t^{16}+1)}{t^2+1}\, dt = \pi \ln[\ 2^4 (1 + \sin \tfrac{\pi}{16})\ (1 + \sin \tfrac{3\pi}{16})\ (1 + \sin \tfrac{5\pi}{16})\ (1 + \sin \tfrac{7\pi}{16})] \qquad Eq.9.6.$$

This pattern continues with higher values of n. The highest value for which I have checked the analytical result with numerical quadrature is $n = 8$, for which

$$\int_0^\infty \frac{\ln(t^{32}+1)}{t^2+1}\, dt = \pi \ln[\ 2^8 (1 + \sin \tfrac{\pi}{32})\ (1 + \sin \tfrac{3\pi}{32})\ \cdots\ (1 + \sin \tfrac{15\pi}{32})] = 29.336572 \qquad Eq.9.7,$$

the product being over odd values of $\frac{m\pi}{32}$ from $m = 1$ to $m = 2n - 1$.

9.3 General results for $\ln(t^{4n} + 1)$

In general

$$t^{4n} + 1 = \prod_{m=1, m \text{ odd}}^{2n-1} \left[(t^2+1)^2 - 4t^2 \cos^2 \left(\frac{m\pi}{4n} \right) \right],$$

The factorisation works because of the pairwise simplifications for all m, n, A, B

$$\cos^2 \left[\frac{m\pi}{4n} \right] + \cos^2 \left[\frac{(2n-m)\pi}{4n} \right] = 1 \quad \text{and}$$

$$\cos[A\pi] \cos[B\pi] + \cos[(2n-A)\pi] \cos[(2n-B)\pi] = \cos[(B-A)\pi]$$

together with the identity

$$\prod_{m=1, m \text{ odd}}^{2n-1} \cos \left(\frac{m\pi}{4n} \right) = 2^{2n-1}.$$

The resulting integral is

$$\int_0^\infty \frac{\ln(t^{4n}+1)}{t^2+1}\,dt = \pi \ln\left[2^n \prod_{m=1,m \text{ odd}}^{2n-1} \left[1+\sin\left(\frac{m\pi}{4n}\right)\right]\right]. \qquad Eq.9.8$$

As explained in §8.2 the significance of this explicit formula is that $x^{2^n}+1$ is an irreducible factor of x^p+1 for many values of integer p.

9.4 General results for descending sequence polynomials

The cases of $k=6$ and $k=10$ in §9.2 give a clue to the more general way of obtaining a factorisation of $x^{2(p-1)} - x^{2(p-2)} + \cdots + x^4 - x^2 + 1$ for p odd. When p is an odd prime, these are factors in $x^{2p}+1$ (see §8.2). Using the notation $F(k) = (t^2+1)^2 - 4t^2\cos^2\frac{\pi}{k}$ introduced with Eq. 10, the family of alternating descending series polynomials commences

$$\begin{aligned}
F(\tfrac{6}{1}) &= t^4 - t^2 + 1\\
F(\tfrac{10}{1})F(\tfrac{10}{3}) &= t^8 - t^6 + t^4 - t^2 + 1\\
F(\tfrac{14}{1})F(\tfrac{14}{3})F(\tfrac{14}{5}) &= t^{12} - t^{10} + t^8 - t^6 + t^4 - t^2 + 1\\
F(\tfrac{18}{1})F(\tfrac{18}{3})F(\tfrac{18}{5})F(\tfrac{18}{7}) &= t^{16} - t^{14} + t^{12} - t^{10} + t^8 - t^6 + t^4 - t^2 + 1\\
F(\tfrac{22}{1})F(\tfrac{22}{3})F(\tfrac{22}{5})F(\tfrac{22}{7})F(\tfrac{22}{9}) &= t^{20} - t^{18} + t^{16} - \cdots + t^8 - t^6 + t^4 - t^2 + 1\\
F(\tfrac{26}{1})F(\tfrac{26}{3})F(\tfrac{26}{5})F(\tfrac{26}{7})F(\tfrac{26}{9})F(\tfrac{26}{11}) &= t^{24} - t^{22} + t^{20} - \cdots - t^6 + t^4 - t^2 + 1.
\end{aligned}$$

The related integral is

$$\int_0^\infty \frac{\ln(t^{4n} - t^{4n-2} + t^{4n-4} - \cdots - t^2 + 1)}{t^2+1}\,dt = \pi \ln\left[2^n \prod_{m=1,m \text{ odd}}^{2n-1} \left[1+\sin\left(\frac{m\pi}{4n+2}\right)\right]\right]. \qquad Eq.9.9$$

Note in passing that shifting the values of k to even denominators gives the corresponding non-alternating descending series. For example (see case $k=5$ in §9.2)

$$F(\tfrac{10}{2})F(\tfrac{10}{6}) = t^8 + t^6 + t^4 + t^2 + 1$$

$$F(\tfrac{14}{2})F(\tfrac{14}{4})F(\tfrac{14}{6}) = t^{12} + t^{10} + t^8 + t^6 + t^4 + t^2 + 1.$$

These factorise into products of other descending series –

$$x^8 + x^6 + x^4 + x^2 + 1 = (x^4 - x^3 + x^2 - x + 1)(x^4 + x^3 + x^2 + x + 1)$$

$$x^{12} + x^{10} + x^8 + x^6 + x^4 + x^2 + 1 = (x^6 - x^5 + x^4 - x^3 + x^2 - x + 1)(x^6 + x^5 + x^4 + x^3 + x^2 + x + 1)$$

From §8.2 we recognise one factor on the right as a factor of $x^p+1 = (x+1)(x^{p-1} - x^{p-2} + \cdots + x^2 - x + 1)$ for p an odd prime. Unfortunately, I cannot see how to isolate this factor, and so cannot see how this approach could yield integrals involving x^p+1 for p an odd prime. For these cases a different approach to the integration seems necessary. Similarly I have been unable to use this method to find integrals involving descending series of the type $x^{2j} - x^j + 1$ for any integer other than $j = 2^n$ (*e.g.* case $k=12$ in §8.2). These particular cases are of no additional value since they occur only in the factorisation of $\int_0^\infty \ln(t^{4n}+1)/(t^2+1)\,dt$, for which I have given a formula in Eq. 9.8.

9.5 Results for Putnam-type integrals involving $\ln(x^p + 1)$*, formed by combining factors*

Here I collect together the results from above which can be used, together with the factorisations in §8.2, to evaluate a few integrals for further values of p.

$$\int_0^\infty \frac{\ln(t+1)}{t^2+1}dt = \frac{\pi}{4}\ln 2 + \mathbf{G} \quad , \qquad \int_0^\infty \frac{\ln(t^2+1)}{t^2+1}dt = \pi \ln 2.$$

$$\int_0^\infty \frac{\ln(t^4 - t^2 + 1)}{t^2+1}dt = \pi \ln 3 \quad , \qquad \int_0^\infty \frac{\ln(t^8 - t^6 + t^4 - t^2 + 1)}{t^2+1}dt = \pi \ln(5 + 2\sqrt{5})$$

$$\int_0^\infty \frac{\ln(t^{12} - t^{10} + t^8 - t^6 + t^4 - t^2 + 1}{t^2+1)}dt = \pi \ln[\ 8(1 + \sin\tfrac{\pi}{14})\ (1 + \sin\tfrac{3\pi}{14})\ (1 + \sin\tfrac{5\pi}{14})]$$

Combining pairs of these we obtain

$$\int_0^\infty \frac{\ln(t^6+1)}{t^2+1}dt = \pi(\ln 2 + \ln 3) = \pi \ln 6.$$

$$\int_0^\infty \frac{\ln(t^{10}+1)}{t^2+1}dt = \pi \ln(10 + 4\sqrt{5}).$$

$$\int_0^\infty \frac{\ln(t^{14}+1)}{t^2+1}dt = \pi \ln[\ 16(1 + \sin\tfrac{\pi}{14})\ (1 + \sin\tfrac{3\pi}{14})\ (1 + \sin\tfrac{5\pi}{14})] \qquad Eq.9.10$$

and similarly for $p = 2\times$(odd prime). This completes the table in §8.1 as far as this method allows.

9.6 Putnam-type integrals over $[0, 1]$ *derived from integrals over* $[0, \infty)$

Divide the limits of integration into two parts, one from 0 to 1 and the other from 1 to ∞ in the same way as was done in discussing Catalan's constant, $\mathbf{G}$, in §5. If p in any in real number > 1 (it does not have to be an integer)

$$\int_0^\infty \frac{\ln(x^p+1)}{x^2+1}dx = \int_0^1 \frac{\ln(x^p+1)}{x^2+1}dx + \int_1^\infty \frac{\ln x^p}{x^2+1}dx + \int_1^\infty \frac{\ln(1+\frac{1}{x^p})}{x^2+1}dx.$$

The second integral on the right is $p\mathbf{G}$. The third integral on the right is identical to the first on the right, as can be seen by making the substitution $1/x = u$, $dx = -du/u^2$,

$$\int_1^\infty \frac{\ln(1+\frac{1}{x^p})}{x^2+1}dx = \int_1^0 \frac{\ln(1+u^p)}{\frac{1}{u^2}+1}\left(\frac{-du}{u^2}\right) = \int_0^1 \frac{\ln(1+u^p)}{u^2+1}du$$

We have established for all real $p > 1$ that

$$\int_0^\infty \frac{\ln(x^p+1)}{x^2+1}dx = 2\int_0^1 \frac{\ln(x^p+1)}{x^2+1}dx + p\mathbf{G}. \qquad Eq.9.11$$

The integral on the right is the Putman challenge problem with x generalised to x^p.

Table 9.1 below is derived from Table 8.1, §8.1, by using the above formula.

Table 9.1: Values of the generalised Putnam integral between limits 0 to 1

p	$\int_0^1 \frac{\ln(x^p+1)}{x^2+1}dx$	formula
0	0·544396	$\frac{\pi}{4}\ln 2 = \ln 2 \tan^{-1}(1)$
$\frac{1}{2}$	0·370346	
1	0·272198	$\frac{\pi}{8}\ln 2$
$\frac{3}{2}$	0·212341	
2	0·172827	$\frac{\pi}{2}\ln 2 - \mathbf{G}$
$\frac{5}{2}$	0·145098	
3	0·124704	Eq. 12.7
$\frac{7}{2}$	0·109146	
4	0·096924	$\frac{\pi}{2}\ln(2+\sqrt{2}) - 2\mathbf{G}$
$\frac{9}{2}$	0·087091	
5	0·079022	
$\frac{11}{2}$	0·072290	
6	0·066592	$\frac{\pi}{2}\ln 6 - 3\mathbf{G}$
$\frac{13}{2}$	0·061712	
7	0·057487	
8	0·050544	$\frac{\pi}{2}\ln(4+\sqrt{2}+2\sqrt{2+\sqrt{2}}+2\sqrt{2-\sqrt{2}}) - 4\mathbf{G}$
9	0·045080	
10	0·040672	$\frac{\pi}{2}\ln(10+4\sqrt{5}) - 5\mathbf{G}$
12	0·034005	$\frac{\pi}{2}\ln\,(4\sqrt{6}+4\sqrt{3}+5\sqrt{2}+10\ ) - 6\mathbf{G}$
14	0·029206	see Eq. 9.9
16	0·025589	see Eq. 9.6
20	0·020503	Eq. 9.8 with $n=5$

9.7 Some related integrals over domain $-\infty \to +\infty$

Refer back to §9.1 and this time consider the contour segment along the real axis, previously denoted by γ_1 and γ_3, as a *single* segment parameterised by t with $-\infty < t < +\infty$. Then Eq. 9.1 a and b together with Eq 9.2 give

$$\int_{-\infty}^{\infty} \frac{\ln(t^2 \pm 2t\cos\frac{\pi}{k}+1)}{t^2+1}dt = \pi\ln(2+2\sin\tfrac{\pi}{k})$$

Selected simple values of k give the results on the next page.

We would really like $\int \ln(t^2 \pm t + 1)/(t^2+1)dt$ from 0 to ∞ because it is a factor in the integral of $\ln(x^3+1)$, but for now we will have to be content with the much less interesting and less useful integral from $-\infty$ to $+\infty$.

$$k = \begin{cases} 0 \text{ or } \pi : & \displaystyle\int_{-\infty}^{\infty} \frac{\ln(t^2 \pm 2t + 1)}{t^2+1} dt = 2\int_{-\infty}^{\infty} \frac{\ln(t \pm 1)}{t^2+1} dt = \pi \ln 2 \\ \frac{\pi}{2} \text{ or } \frac{3\pi}{2} : & \displaystyle\int_{-\infty}^{\infty} \frac{\ln(t^2+1)}{t^2+1} dt = \pi \ln 2 \\ \frac{\pi}{3} \text{ or } \frac{2\pi}{3} : & \displaystyle\int_{-\infty}^{\infty} \frac{\ln(t^2 \pm t + 1)}{t^2+1} dt = \pi \ln(2+\sqrt{3}) \\ \frac{\pi}{4} \text{ or } \frac{3\pi}{4} : & \displaystyle\int_{-\infty}^{\infty} \frac{\ln(t^2 \pm \sqrt{2}t + 1)}{t^2+1} dt = \pi \ln(2+\sqrt{2}) \\ \frac{\pi}{6} \text{ or } \frac{5\pi}{6} : & \displaystyle\int_{-\infty}^{\infty} \frac{\ln(t^2 \pm \sqrt{3}t + 1)}{t^2+1} dt = \pi \ln 3 \end{cases}$$

10. Integrals of rational functions with $\ln(x^3+1)$ which have primitives

In §2 I pondered the fact that most integrals do not have a closed form solution. So far in this paper I have managed to evaluate definite integrals of $\ln(x^p+1)/(x^2+1)$ where p is a positive integer equal to:

a) 1 or 2
b) any integer multiple of 4 (4, 8, 12, 16, 20, 24, etc.)
c) twice an odd prime (6, 10, 14, 22, 26, etc.)

All other cases remain open for further investigation. While my target has been to find analytical answers for

$$\int_0^{\infty} \frac{\ln(x^p+1)}{x^2+1}\, dx \text{ for } p \text{ an odd prime}$$

(assuming they exist!), I have been forced to recognise that so many integrals involving $\ln(x^p+1)$ times any rational function are intractable. I have therefore settled for trying to find even a few integrals for the lowest case $p = 3$. There is some hope because in §4.7 quotes four from Gradshteyn and Ryzhik's book, and in §11 I show how to evaluate some of these.

10.1 Search for primitives of integrals of rational functions and $\ln(x^3+1)$

It is natural to start any problem in integration by looking for primitives – that is, by performing the indefinite integration. I have used the symbolic maths software Mathematica, Macsyma and Reduce to search a sample of integrands composed of a rational function $f(x)/g_n(x)$ multiplied by $\ln(x^3+1)$. Here the denominator $g_n(x)$ is indexed by an integer, n, related to its degree as explained below. The least complicated primitive I came across is

$$\int \frac{x^2 \ln(x^3+1)}{(x^3+1)^2}\, dx = -\frac{1+\ln\left(x^3+1\right)}{3\left(x^3+1\right)}.$$

I found very few primitives in cases where the denominator has low degree, but for n greater than a threshold value, some sequences over n of $g_n(x)$ did have primitives – composed of complicated sums of rational functions, inverse tangents and logarithms. Here are some examples:

$$\text{a)} \int \frac{x \ln\left(x^3+1\right)}{\left(x^2+1\right)^n}\, dx$$

has a primitive for all the positive integer values of $n \geq 2$ which I tried (but $n = 1$ has no primitive). As an example with $n = 2$

$$\int \frac{x \ln\left(x^3+1\right)}{\left(x^2+1\right)^2}\, dx = \frac{\ln\left(x^3+1\right)}{4} - \frac{3}{8}\ln\left(x^2+1\right) + \frac{\sqrt{3}}{2}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) - \frac{3}{4}\tan^{-1} x - \frac{\ln\left(x^3+1\right)}{2\left(x^2+1\right)}.$$

There are also primitives for the sequences

$$\text{b)} \int \frac{\ln\left(x^3+1\right)}{(x+1)^n}\,dx \text{ for } n \geq 2.$$

$$\text{c)} \int \frac{x\,\ln\left(x^3+1\right)}{(x+1)^n}\,dx \text{ for } n \geq 3.$$

$$\text{d)} \int \frac{x^2\,\ln\left(x^3+1\right)}{(x+1)^n}\,dx \text{ for } n \geq 4.$$

On the other hand none of the four sequences

$$\int \frac{\ln\left(x^3+1\right)}{\left(x^2+1\right)^n}\,dx\ ,\ \int \frac{x^2\,\ln\left(x^3+1\right)}{\left(x^2+1\right)^n}\,dx,$$

$$\int \frac{\ln\left(x^3+1\right)}{\left(x^3+1\right)^n}\,dx \text{ and } \int \frac{x\,\ln\left(x^3+1\right)}{\left(x^3+1\right)^n}\,dx$$

seems to have a primitive for any positive integer n. I have not investigated why some sequences of functions do have primitives, but it seems likely that the integral for n can be expressed in terms of the integral for $n-1$, in a chain down to some least value of n.

The challenge before us is to evaluated (in algebraic form) at least one integral involving $\ln(x^3+1)$ times a rational function for which a primitive does *not* exist.

10.2 A 'limiting thin wedge contour' method for $\oint \ln(z^3+1)F(z)\,dz$ for $F(z)$ a rational function
The 'method' I describe in this section arose from an attempt to evaluate integrals of this type. The argument of the logarithm has three zeros, where $z^3=-1$ and at each of these three points the logarithm is negatively infinite in value and, moreover, any contour encircling one of these branch points must jump in value by $2\pi\imath$. Hence it is necessary to choose a contour which avoids these points and the three branch cuts which radiate from them. This led me to consider a wedge contour along the real axis, extending to $+\infty$ but not extending far enough into the upper half plane to encroach on the branch cut from $z=e^{\imath\pi/3}$.

With this motivation in mind, choose the wedge contour shown in Figure 10.1 below. The segment γ_1 runs along the positive x axis to R, which is taken to its limit at infinity. γ_2 is a small arc which contributes nothing to the integral as $R \to \infty$ provided – and (as in §6) this is crucial – that $z\,F(z) \to 0$ as $z \to \infty$. The segment γ_3 returns to the origin to form a wedge which is narrow enough for z^3+1 to avoid the branch cuts emanating from $z=e^{\imath\pi/3}$ and $z=e^{5\imath\pi/3}$. That means that the wedge angle β is less that $\pi/3$.

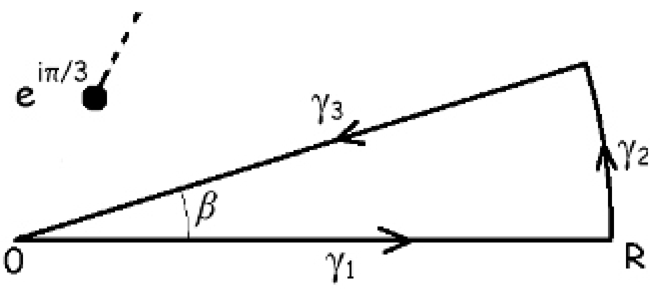

Figure 10.1: Thin wedge contour avoiding branch cuts

Regard β as a parameter and develop the real and imaginary parts of the contour integral as a Taylor series in β, retaining only terms up to, say, β^2. The loop value of the real and imaginary parts will each be zero (or a constant in the unusual case that there is a pole on the positive real axis.) Since the power series evaluates to zero, or the same constant, for all small values of β, the coefficients of the power series must themselves be zero. By isolating the coefficient of each power of β in turn we find relationships between integrals over rational functions and logarithms. In particular, the contributions to coefficients of β^0, coming from the segments γ_1 and γ_3 will in most cases cancel. The coefficients of β^1 provide the most useful information. They are formed from two or more terms, some of which contain unknown integrals over logarithms but others consist only of rational functions; for these the definite integral can be found by techniques such as those of Copson in §6. I illustrate it by an example.

Let

$$F(z) = \frac{\ln\left(z^3+1\right)}{z^3+1}.$$

This has no singularities along the positive real axis. Along segment γ_1 $F(z) = \ln\left(t^3+1\right)/\left(t^3+1\right)$. The contribution from γ_2 is zero. Along the returning segment of the wedge, γ_3, $z = te^{\imath\beta}$, $dz = e^{\imath\beta}\,dt$. Taking account of the direction of travel around the closed contour, the loop integral is

$$\oint F(z)\,dz = \int_0^\infty \left[F(t) - e^{\imath\beta}F(te^{\imath\beta})\right]\,dt = 0$$

Now expand this as a Taylor series in β about $\beta = 0$. This is most conveniently done using a symbolic maths package such as Macsyma, Maxima or Mathematica, or by differentiation with respect to β. However, we only want the first two terms and these can be obtained 'by hand' using the limiting approximations

$$\sin\beta \to \beta\ ,\ \cos\beta \to 1\ ,\ \tan^{-1}(k\beta) \to k\beta$$

for a constant k. Also, expand the denominator by the binomial theorem. Explicitly the integrand on γ_3 is

$$\begin{aligned} -\frac{e^{\imath\beta}\,\ln\left(t^3e^{3\imath\beta}+1\right)}{t^3e^{3\imath\beta}+1} &\approx (1+\imath\beta)\left\{\ln(t^3+1) + \imath\left(\frac{3\beta t^3}{t^3+1}\right)\right\}\frac{1}{t^3+1}\left\{1-\imath\frac{3\beta t^3}{t^3+1}\right\} \\ &\approx -\frac{\ln\left(t^3+1\right)}{t^3+1} + \imath\beta\frac{\left(2\,t^3-1\right)\,\ln\left(t^3+1\right)-3t^3}{(t^3+1)^2} \end{aligned}$$

to terms in β^1. It is clear that the term in β^0 exactly cancels the integrand along γ_1 and that the coefficient of β^1 is purely imaginary. From this

$$\int_0^\infty \frac{\left(2\,t^3-1\right)\,\ln\left(t^3+1\right)}{(t^3+1)^2}\,dt = 3\int_0^\infty \frac{t^3}{(t^3+1)^2}\,dt\,. \qquad Eq.10.1$$

Now the right hand side can be evaluated by the methods of §6 and its value is $2\pi\sqrt{3}/9 = 1\cdot 20920$ so, in so far as it is useful, we have obtained the value of the integral involving $\ln(t^3+1)$ on the left.

The result, however, is somewhat disappointing because a) the integrand on the left of Eq. 10.1 has two terms so is too complicated and particular to be of much wider interest, b) the integral has a primitive, which is

$$-\frac{\ln\left(t^2-t+1\right)}{6} + \frac{\tan^{-1}\left(\frac{2\,t-1}{\sqrt{3}}\right)}{\sqrt{3}} + \frac{\ln(t+1)}{3} - \frac{t}{t^3+1} - \frac{t\,\ln\left(t^3+1\right)}{t^3+1}\,.$$

Interestingly neither of its two constituent terms has a primitive. The degree of the numerator on the left side of Eq. 10.1 can be reduced by changing the variable in the integral with the t^3 factor by the substitution $t = 1/u$, $dt = -du/u^2$. This gives

$$\int_0^\infty \frac{t^3 \ln\left(t^3+1\right)}{(t^3+1)^2}\, dt = \int_0^\infty \frac{t \ln\left(t^3+1\right)}{(t^3+1)^2}\, dt - 3\int_0^\infty \frac{t \ln t}{(t^3+1)^2}\, dt$$

and again the second integral on the right can be evaluated by the key-hole contour method on §6.7. Its value is

$$\int_0^\infty \frac{t \ln t}{(t^3+1)^2}\, dt = -\frac{2\sqrt{3}\pi}{27} + \frac{2\pi^2}{81} = -0\cdot 1593726\ .$$

Putting this together with Eq. 10.1 gives

$$\int_0^\infty \frac{(2t-1)\, \ln\left(t^3+1\right)}{(t^3+1)^2}\, dt = \frac{4\pi^2}{27} - \frac{2\pi\sqrt{3}}{9} = 0\cdot 252964. \qquad Eq.10.2$$

This integral does not have a primitive.

Ideally, we would like to separate the two terms which make up this integral and evaluate each constituent separately. To this end I have tried other rational functions in the integrand. For example $F(z) = z\ln(z^3+1)/(z^3+1)$ takes us on a journey of calculation round the thin wedge contour similar to the above, and the substitution $t = 1/u$ again reduces the degree of one term –

$$\int_0^\infty \frac{t^4 \ln\left(t^3+1\right)}{(t^3+1)^2}\, dt = \int_0^\infty \frac{\ln\left(t^3+1\right)}{(t^3+1)^2}\, dt - 3\int_0^\infty \frac{\ln t}{(t^3+1)^2}\, dt.$$

We encounter

$$\int_0^\infty \frac{\ln t}{(t^3+1)^2}\, dt = -\frac{2\pi\sqrt{3}}{27} - \frac{4\pi^2}{81} \quad \text{and} \quad \int_0^\infty \frac{t^4}{(t^3+1)^2}\, dt = \frac{4\pi\sqrt{3}}{27},$$

but, after some cancellation, the journey arrives at exactly the result of Eq. 10.2. A little progress towards expressing Eq. E in terms of more elementary integrals can be achieved by splitting the rational factor in the integrand into partial fractions in the form

$$\frac{2t-1}{(t^3+1)^2} = \frac{t-3}{3(t^3+1)} + \frac{-t^4+3t^3+5t}{3(t^3+1)^2}.$$

The degree of the terms in t^4 and t^3 can be reduced, as above, by the substitution $t = 1/u$, and so Eq. 10.2 can be recast in the form

$$\int_0^\infty \frac{t\ln(t^3+1)}{t^3+1}\, dt - 3\int_0^\infty \frac{\ln(t^3+1)}{t^3+1}\, dt + 3\int_0^\infty \frac{\ln(t^3+1)}{(t^3+1)^2}\, dt = \frac{2\pi}{9}(\pi - \sqrt{3}) = 0\cdot 984046. \qquad Eq.10.3$$

The first two integrals were quoted in §4.7, Eq. 4.3d, 4.3e from Gradshteyn and Ryzhik's book, and are derived in detail in §12. Substituting for these, we can solve for

$$\int_0^\infty \frac{\ln(t^3+1)}{(t^3+1)^2}\, dt = \frac{2\,\pi}{27}\left(3\sqrt{3}\,\log 3 - \pi - \sqrt{3}\right) = 0\cdot 19429318\ .$$

10.3 Critique of the 'limiting thin wedge' method

This method is most readily applied using symbolic maths software such as Maxima or Mathematica since the algorithm is as follows

Step 1: Define a rational function $F(z) = f(z)/g(z)$ such that $\lim_{z\to\infty} z\,F(z) = 0$

Step 2: Define the complex integrand $H(z) = F(z)\ln(x^p + 1)$ for some integer p (here 3)

Step 3; construct the function $L(t) = H(t) - e^{\imath\beta}H(te^{\imath\beta})$ which, when integrated from 0 to ∞, is the value of the loop integral round the wedge.

Step 4: Expand $L(t)$ as a Taylor series in β about $\beta = 0$ and take the coefficient of β^1.

Step 5: Equate the real and imaginary parts of this coefficient respectively to zero. The general pattern of outcome is that

$$\oint \frac{f(z)\ln(z^p+1)}{g(z)}\,dz \quad \text{gives rise to} \quad \int_0^\infty \frac{A(x)\ln|x^p+1|}{B(x)}\,dx = \int_0^\infty \frac{C(x)}{D(x)}\,dx$$

where A, B, C and D are polynomials.

Step 6: Evaluate the integral of $R(x) = C(x)/D(x)$ by the methods of §6.6 and/or §6.7.

The limitations of the method are that :

a) the denominator $B(x)$ is generally a complicated polynomial. Moreover, unless there are fortuitous cancellations, it is always a perfect square and as such cannot give the denominator x^2+1 required for Putnam-type integrals. However, there may be some mileage in splitting the A/B into partial fractions.

b) the polynomial factor $A(x)$ in the numerator implies a sum of terms which cannot readily be separated.

c) it is difficult to predict and hence control the resulting real integral. There may be an art in selecting the rational function $F(z)$ suitable for creating the desired real integral.

We can gain a deeper insight into the constitution of the polynomials A, B, C and D by considering the nature of the Taylor series expansion. To be quite general I use the notation that the complex integrand on the contour is $H(z) = f(z)\ln h(z)/g(z)$ (where $h(z)$ means z^3+1), the Taylor expansion about $\beta = 0$ on the γ_3 segment is

$$e^{\imath\beta}H(t,\beta) = e^{\imath\beta}\left[H(t,0) + \beta\left.\frac{\partial H}{\partial\beta}\right|_{\beta=0} + \frac{\beta^2}{2!}\left.\frac{\partial^2 H}{\partial\beta^2}\right|_{\beta=0} + \cdots\right]$$

The contribution from segment γ_1 is precisely $H(t,0)$, which cancels the first term above as $\beta \to 0$, while the coefficient of β is

$$\frac{1}{g^2 h}\left[(gf' - fg' + \imath fg)\,h\ln h + fgh'\right]\Big|_{\beta=0}.$$

where $'$ denotes differentiation with respect to β. Since this coefficient is zero, we obtain the equation of integrals

$$\int_0^\infty \frac{(fg' - gf' - \imath fg)}{g^2}\ln h\,dx = \int_0^\infty \frac{f}{g}\frac{h'}{h}\,dx$$

which accounts for A, B, C and D, subject to cancellation of any common factors. This formula is quite general and could in principle be used to investigate integrals involving a wide range of rational, trigonometric, exponential and other functions multiplied by a logarithm. However, if we restrict for the time being our concern to $h(z) = z^3 + 1$, then $h'/h = 3x^3/(x^3 + 1)$ as $\beta \to 0$.

If we also restrict f and g to be polynomials in $z = te^{\imath\beta}$, it is possible to select the coefficients of these polynomials in order to control to some limited extent the rational function which multiplies the logarithm. I say 'to some limited extent' because there are only a few coefficients to specify, bearing in mind that the degree of g must be at least 2 greater than the degree of f in order to meet the condition for vanishing at infinity. A simple case is $f = 1$, $g = z^2$. This leads to the equation of integrals

$$\int_0^\infty \frac{\ln(x^3+1)}{x^2}\,dx = 3\int_0^\infty \frac{x}{x^3+1}\,dx = \frac{2\pi}{\sqrt{3}} = 3\cdot 6276.$$

This would be a welcome novel result if the integral on the left did not have a primitive! Nevertheless, it is correct. The result comes about purely because the upper limit of integration is infinity. The respective integrals to any finite limit will differ, but the difference term tends to zero at infinity.

The each case integration of the rational function can be performed by the key-hole contour method of §6.6, or by finding its primitive. Regrettably, I find it impossible by this approach to create the integral of $\ln(x^3 + 1)/(x^2 + 1)\,dx$, which remains an elusive prize. Here are a few examples of equations involving $\ln(x^3 + 1)$ which have been generated:

1) $$\oint \frac{\ln(z^3+1)}{z^2}\,dz \quad \text{leads to} \quad \int_0^\infty \frac{\ln(x^3+1)}{x^2}\,dx = \int_0^\infty \frac{3x}{x^3+1}\,dx = \frac{2\pi}{\sqrt{3}} = 3\cdot 6276.$$

2) $$\oint \frac{\ln(z^3+1)}{z(z+1)}\,dz \text{ leads to } \int_0^\infty \frac{\ln\left(x^3+1\right)}{(x+1)^2}\,dx = \int_0^\infty \frac{3x^2}{(x+1)(x^3+1)}\,dx = \frac{2\pi\sqrt{3}}{9} + 1 = 2\cdot 2092$$

3) $$\oint \frac{\ln(z^3+1)}{z^2+1}\,dz \text{ leads to } \int_0^\infty \frac{(x^2-1)\ln(x^3+1)}{(x^2+1)^2}\,dx = \int_0^\infty \frac{3x^3}{(x^2+1)(x^3+1)}\,dx = \frac{3\pi}{4} = 2\cdot 3562$$

4) $$\oint \frac{\ln(z^3+1)}{z(z^2+1)}\,dz \text{ leads to } \int_0^\infty \frac{x\ln(x^3+1)}{(x^2+1)^2}\,dx = \int_0^\infty \frac{3x^2}{2(x^2+1)(x^3+1)}\,dx = (8\sqrt{3}-9)\frac{\pi}{24} = 0\cdot 6357$$

5) $$\oint \frac{\ln(z^3+1)}{z^3}\,dz \text{ leads to } \int_0^\infty \frac{\ln(x^3+1)}{x^3}\,dx = \int_0^\infty \frac{3}{2(x^3+1)}\,dx = \frac{\pi}{\sqrt{3}} = 1\cdot 8138$$

6) $$\oint \frac{z\ln(z^3+1)}{z^3+1}\,dz \text{ leads to } \int_0^\infty \frac{(x^4-2x)\ln(x^3+1)}{(x^3+1)^2}\,dx = \int_0^\infty \frac{3x^4}{(x^3+1)^2}\,dx = \frac{4\pi\sqrt{3}}{9} = 2\cdot 4184$$

7) $$\oint \frac{\ln(z^3+1)}{z^3+1}\,dz \text{ leads to } \int_0^\infty \frac{(2x^3-1)\ln(x^3+1)}{(x^3+1)^2}\,dx = \int_0^\infty \frac{3x^3}{(x^3+1)^2}\,dx = \frac{2\pi\sqrt{3}}{9} = 1\cdot 2092$$

8) $$\oint \frac{\ln(z^3+1)}{z(z^3+1)}\,dz \text{ leads to } \int_0^\infty \frac{x^2\ln(x^3+1)}{(x^3+1)^2}\,dx = \int_0^\infty \frac{x^2}{(x^3+1)^2}\,dx = \frac{1}{3}$$

9) $$\oint \frac{\ln(z^3+1)}{z^2(z^3+1)}\,dz \text{ leads to } \int_0^\infty \frac{(4x^3+1)\ln(x^3+1)}{x^2(x^3+1)^2}\,dx = \int_0^\infty \frac{3x}{(x^3+1)^2}\,dx = \frac{2\pi\sqrt{3}}{9} = 1\cdot 2092$$

10) $$\oint \frac{\ln(z^3+1)}{z^3(z^3+1)}\,dz \text{ leads to } \int_0^\infty \frac{(5x^3+2)\ln(x^3+1)}{x^3(x^3+1)^2}\,dx = \int_0^\infty \frac{3}{(x^3+1)^2}\,dx = \frac{4\pi\sqrt{3}}{9} = 2\cdot 4184$$

I can see no *a priori* reason why the integrals $\int_0^\infty$ on the left, involving $\ln(x^3+1)$, should have a primitive with respect to x, but every one of the functions above *does* have a primitive. Speaking heuristically, it as if the limiting wedge contour is insensitive to the critical, essential features of those types of log integrals which do *not* have a primitive. To explain this a little further, note that integrals 3), 6), 7), 9) and 10) above have two terms in the numerator $A(x)$. Neither of the two individual, elementary integrals defined by this numerator has a primitive – only their sum or difference, as shown. Take 3) as an example: numerical quadrature gives

$$\int_0^\infty \frac{x^2 \ln(x^3+1)}{(x^2+1)^2}\,dx = 2\cdot 67674966 \quad \text{and} \quad \int_0^\infty \frac{\ln(x^3+1)}{(x^2+1)^2}\,dx = 0\cdot 32055517$$

and neither of these has a primitive. Their difference is $2\cdot 3561945 = 3\pi/4$ as given above. Interestingly, their sum is $2\cdot 99730483$ which can be identified with the elusive Putnam-type integral $\int_0^\infty \frac{\ln(x^3+1)}{x^2+1}\,dx$. This happens because the partial fraction decomposition of

$$\frac{x^2}{(x^2+1)^2} \quad \text{is} \quad \frac{1}{x^2+1} - \frac{1}{(x^2+1)^2}.$$

So each of the two individual elementary integrals must have a term which cancels on subtraction – possibly a term which can only be represented by an infinite series.

One might wonder whether the higher terms in the Taylor series of the contour integral, involving higher derivates of the integrand, furnish extra detail. I have not found this to be the case. The second derivatives typically involve higher powers of the denominator. For example, the second derivative of case 3) above $\oint \ln(z^3+1)/(z^2+1)dz$ leads to the equation of integrals

$$\int_0^\infty \frac{(x^4-6x^2+1)\ln(x^3+1)}{(x^2+1)^3}\,dx = 3\int_0^\infty \frac{-2x^8+2x^6+x^5+5x^3}{(x^2-x+1)^2(x^2+1)^2(x+1)^2}\,dx.$$

The right hand side, though complicated, can be evaluated by the methods on §6, but the left side has a partial fraction in $\ln(x^3+1)/(x^2+1)^3$ whose value cannot be determined. So higher terms in the Taylor series do not seem fruitful as a way of evaluating integrals with single term, low order rational functions. But conversely they would allow more complicated integrals such as $\int_0^\infty \ln(x^3+1)/(x^2+1)^3\,dx$ to be built from known simpler ones.

11 Contour integration to evaluate $\int_0^\infty \frac{\ln(x^p+1)}{z^p \pm a^p}\,dx$, $p=3$, etc.

11.1 Contour integration of $\int_0^\infty \frac{\ln(x^3+1)}{z^3-a^3}\,dx$

Here a is a real positive constant. I have chosen this function because it is a non-trivial integral involving $\ln(x^3+1)$ (it does not have a primitive) yet it is less troublesome than either the integrals of $\frac{\ln(x^3+1)}{z^3+a^3}$ or $\frac{\ln(x^3+1)}{z^2 \pm a^2}$, which I tackle later.

We first need to examine the singular points on the integrand, then determine a suitable contour. The poles are at $z=a$, $ae^{2\pi\imath/3}$, $ae^{4\pi\imath/3}$. The argument of the logarithm has zeros at $z=-1$, $e^{\pi\imath/3}$ and $e^{5\pi\imath/3}$, and so these are logarithmic singularities and branch points of the logarithm. The argument of the logarithm would jump by 2π for every turn of z^3+1 around a branch point, so we must make cuts in the complex plane from each of the three branch points as barriers in the path of z. We are free to choose the direction these cuts take. Another consideration in choosing a contour is that it is most convenient if $z^3 + 1$ is real on the main straight segments. The contour I have chosen is shown in Figure 11.1 below.

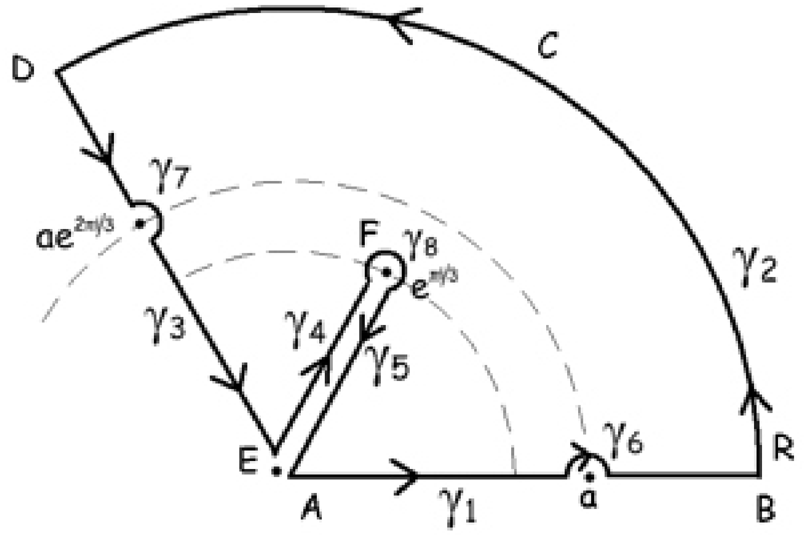

Figure 11.1: Contour for evaluating $\int_0^\infty \frac{\ln(x^3+1)}{x^3-a^3}$

In Figure 11.2 below I have mapped the path of $z^3 + 1$ as z travels round the contour in Figure 11.1. Note the extra 2π in the argument on the segments γ_3 and γ_4.

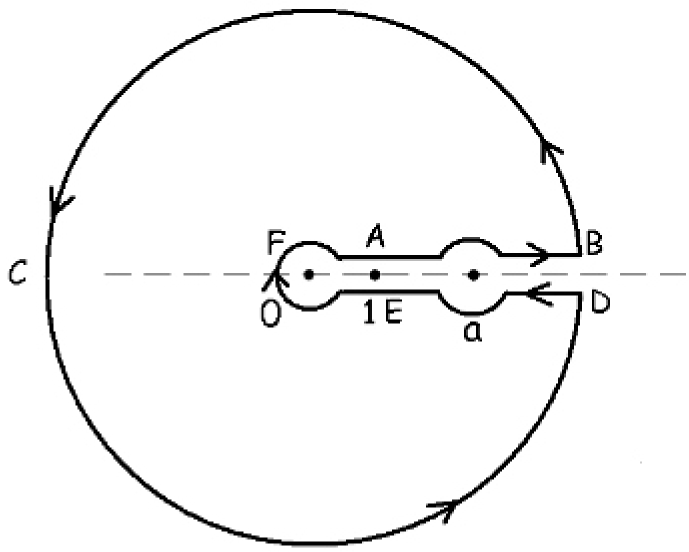

Figure 11.2: path of $z^3 + 1$ as z travels round the contour in Figure 11.1

Now evaluate the contribution on each segment.

γ_1. As the indentation at $z = a$ shrinks to zero, this becomes the Cauchy principal value of the required real integral, $\int_0^\infty \frac{\ln(t^3+1)}{t^3-a^3}\, dt$

γ_2. Here $z = Re^{\imath\theta}$. This contribution tends to zero as the radius R tends to infinity.

γ_3. Here $z = te^{2\pi\imath/3}$. Reference to Figure 11.2 shows that the logarithm has gained an extra 2π in its argument.

$$\int_R^0 \frac{\ln(t^3+1)+2\pi\imath}{t^3-a^3} e^{2\pi\imath/3}\, dt \;\to_{R\to\infty} -\int_0^\infty \frac{\ln(t^3+1)+2\pi\imath}{t^3-a^3}\left(-\tfrac{1}{2}+\imath\tfrac{\sqrt{3}}{2}\right) dt$$

$$= \frac{1}{2}\int_0^\infty \frac{\ln(t^3+1)}{t^3-a^3}\, dt + \pi\sqrt{3}\int_0^\infty \frac{dt}{t^3-a^3}\, dt - \frac{\imath\sqrt{3}}{2}\int_0^\infty \frac{\ln(t^3+1)}{t^3-a^3}\, dt + \imath\pi\int_0^\infty \frac{dt}{t^3-a^3}\, dt$$

where the point $t = a$ is excluded and we understand the integrals to be Cauchy principal values. The first term will combine with the contribution from γ_1.

γ_4. $z = te^{\pi\imath/3}$ and the logarithm still has its extra $2\pi\imath$.

$$\lim_{\varepsilon\to 0}\int_0^{1-\varepsilon} \frac{\ln(-t^3+1)+2\pi\imath}{-t^3-a^3} e^{\imath\pi/3}\, dt$$

γ_5. Here $z = te^{\pi\imath/3}$ again but the logarithm has lost its extra $2\pi\imath$. The direction of integration is opposite to that on γ_4 so, on adding the contributions from γ_4 and γ_5, the log(modulus) parts cancel and leave

$$\int_{\gamma_4} + \int_{\gamma_5} = -2\pi\imath \lim_{\varepsilon\to 0}\int_0^{1-\varepsilon} \frac{e^{\pi\imath/3}}{t^3+a^3}\, dt = \pi(\sqrt{3}-\imath)\int_0^1 \frac{dt}{t^3+a^3}.$$

γ_6. The indentation contributes $-\pi\imath$ times the residue at $z = a$:

$$\int_{\gamma_6} = -\frac{\imath\pi}{3a^2}\ln(a^3+1)$$

γ_7. $z = e^{2\pi\imath/3} + \varepsilon e^{\imath\theta}$ and the logarithm has its extra $2\pi\imath$. The contribution is

$$\lim_{\varepsilon\to 0}\int_{2\pi/3}^{-\pi/3} \frac{\ln[a^3+1-a^2\varepsilon e^{(4\pi\imath/3+\imath\theta)}]+2\pi\imath}{3a^2\varepsilon e^{(4\pi\imath/3+\imath\theta)}}\, \imath\varepsilon e^{\imath\theta}\, d\theta$$

$$= \frac{\pi\sqrt{3}}{6a^2}\ln(a^3+1) - \frac{\pi^2}{3a^2} + \imath\frac{\pi}{6a^2}\ln(a^3+1) + \imath\frac{\pi^2\sqrt{3}}{3a^2}$$

γ_8. This contribution involves the integral over 2π of $\lim_{\varepsilon\to 0}\varepsilon(\ln\varepsilon + K)$ where K is a complex constant. This limit is zero so γ_8 makes zero contribution to the loop integral.

The integral round the closed contour is zero by the residue theorem. Summing the real and imaginary parts we find

$$\text{Real:}\quad \frac{3}{2}\int_0^\infty \frac{\ln(t^3+1)}{t^3-a^3}\, dt + \pi\sqrt{3}\int_0^\infty \frac{dt}{t^3-a^3} + \pi\sqrt{3}\int_0^1 \frac{dt}{t^3+a^3} + \frac{\pi\sqrt{3}}{6a^3}\ln(a^3+1) - \frac{\pi^2}{3a^2} = 0.$$

$$\text{Imag:}\quad \frac{-\sqrt{3}}{2}\int_0^\infty \frac{\ln(t^3+1)}{t^3-a^3}\,dt + \pi\int_0^\infty \frac{dt}{t^3-a^3} - \pi\int_0^1 \frac{dt}{t^3+a^3} - \frac{\pi}{6a^3}\ln(a^3+1) + \frac{\pi^2\sqrt{3}}{3a^2} = 0.$$

If the imaginary part is multiplied by $\sqrt{3}$ and added to the real part, several terms cancel and leave

$$\int_0^\infty \frac{dt}{t^3-a^3} = -\frac{\pi\sqrt{3}}{9a^2},$$

a result which agrees with that in §7 for the case $a=1$. This is substituted back into either the real or imaginary part above.

Another integral term above can be evaluated by partial fraction decomposition; for the case $a=1$

$$\int_0^1 \frac{1}{t^3+1}\,dt \;=\; \frac{1}{3}\int_0^1 \frac{1}{t+1}\,dt - \frac{1}{3}\int_0^1 \frac{t-2}{t^2-t+1}\,dt \;=\; \frac{\ln 2}{3} + \frac{\pi\sqrt{3}}{9},$$

while for a general (real positive) value of a

$$\int_0^1 \frac{1}{t^3+a^3}\,dt \;=\; \frac{\pi\sqrt{3}}{18\,a^2} + \frac{1}{3\,a^2}\ln(a+1) - \frac{1}{6\,a^2}\ln\left(a^2-a+1\right) - \frac{\sqrt{3}}{3\,a^2}\tan^{-1}\left(\frac{a-2}{a\sqrt{3}}\right)$$

Collecting this together we conclude that for $a=1$

$$\int_0^\infty \frac{\ln(x^3+1)}{x^3-1}\,dx = \frac{2\pi^2}{9} - \frac{\pi\ln 2}{\sqrt{3}} = 0\cdot 9360155 \qquad Eq.11.1a$$

and that for a general value of a

$$\int_0^\infty \frac{\ln(x^3+1)}{x^3-a^3}\,dx = \frac{\pi}{3\,a^2}\left(\pi - \sqrt{3}\,\ln(a+1) + 2\,\tan^{-1}\left(\frac{a-2}{a\sqrt{3}}\right)\right). \qquad Eq.11.1b$$

These must be regarded as the Cauchy principal value of the integral through the singularity at $x=a$. The integral for two other values of a is

$$a=2: \quad \frac{\pi}{12}\left(\pi - \sqrt{3}\,\ln 3\right) = 0\cdot 3243015$$

$$a=\tfrac{1}{2}: \quad \frac{4\,\pi}{3}\left(\frac{\pi}{3} - \sqrt{3}\ln\left(\tfrac{3}{2}\right)\right) = 1\cdot 4447614.$$

These agree with values obtained by numerical quadrature.

11.2 Contour integration of $\int_0^\infty \frac{\ln(x^3+1)}{z^3+a^3}\,dx$

This evaluation follows the approach used in §10.2 above with the difference that poles no longer lie in the straight line segments of the contour, γ_1 and γ_3, but instead the pole at $z = a\,e^{\imath\pi/3}$ is aligned with the branch cut of the logarithm. It can have one of three distinct relations to the branch cut:

a) $a>1$: the pole is enclosed by the contour,
b) $a=1$: the pole is superimposed on the logarithmic singularity,
c) $a<1$: the pole lies within the branch cut.

Case b) suggests singular behaviour on the contour at $z=e^{\imath\pi/3}$. However, the integrand is finite and well behaved along the real axis for all values of a, so we must expect any singularities appearing on the complex contour for $a = 1$ to cancel. We shall see that this does indeed happen, though it requires analysis of the limiting behaviour as $a \to 1$.

The contour I have chosen is shown in Figure 11.3 below. The diagram corresponds to $a > 1$.

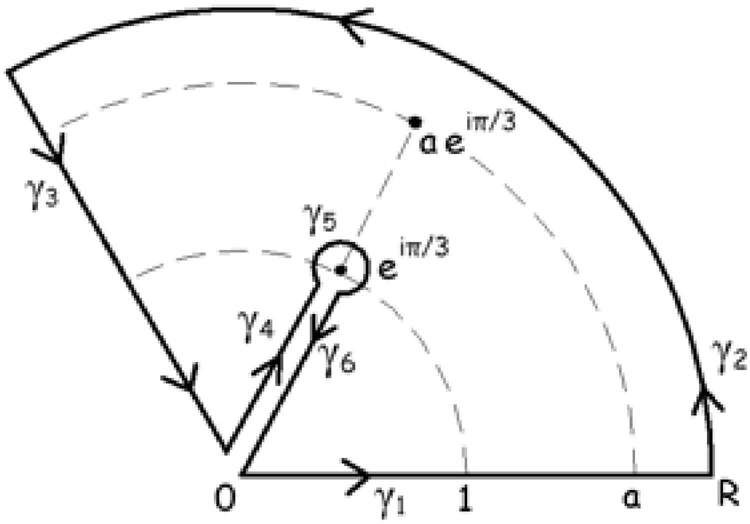

Figure 11.3: contour for evaluating $\int_0^\infty \frac{\ln(x^3+1)}{z^3+a^3}$

Reference to Figure 11.2 will remind the reader of how $z^3 + 1$ travels as z travels round the contour.

The residue at the isolated pole is

$$\left.\frac{\ln(z^3+1)}{3z^2}\right|_{z=a\,e^{\imath\pi/3}} = -\frac{e^{\imath\pi/3}}{3a^2}\left[\ln(a^3-1)+\imath\pi\right]$$

from which the value of the loop integral is found from the residue theorem to be

$$\frac{\pi}{3a^2}(\pi + \sqrt{3}\,\ln(a^3-1)) + \imath\frac{\pi}{3a^2}(\pi\sqrt{3} - \ln(a^3-1)).$$

Now evaluate the contribution from each segment of the contour:

γ_1. This gives the required real integral, $\int_0^\infty \frac{\ln(t^3+1)}{t^3-a^3}\,dt$

γ_2. This contribution tends to zero as the radius R tends to infinity.

γ_3. $z = te^{2\pi\imath/3}$. Reference to Figure 11.2 shows that the logarithm has gained an extra 2π in its argument.

$$\int_R^0 \frac{\ln(t^3+1)+2\pi\imath}{t^3+a^3}\,e^{2\pi\imath/3}\,dt \;\to_{R\to\infty}\; \tfrac{1}{2}\int_0^\infty \frac{\ln(t^3+1)+2\pi\imath}{t^3+a^3}(1-\imath\sqrt{3})\,dt$$

$$= \frac{1}{2}\int_0^\infty \frac{\ln(t^3+1)}{t^3+a^3}\,dt + \pi\sqrt{3}\int_0^\infty \frac{dt}{t^3+a^3}\,dt - \frac{\imath\sqrt{3}}{2}\int_0^\infty \frac{\ln(t^3+1)}{t^3+a^3}\,dt + \imath\pi\int_0^\infty \frac{dt}{t^3+a^3}\,dt$$

As in §10.2 the first term will combine with the contribution from γ_1.

γ_4 plus γ_6. These can be taken together since the log(modulus) parts cancel as in §10.2 and leave

$$\int_{\gamma_4} + \int_{\gamma_6} = 2\pi\imath \lim_{\varepsilon\to 0}\int_0^{1-\varepsilon} \frac{e^{\pi\imath/3}}{-t^3+a^3}\,dt = \pi(-\sqrt{3}+\imath)\int_0^1 \frac{dt}{a^3-t^3}, \qquad a>1$$

Clearly we cannot make the substitution $a=1$; this case is considered separately later.

γ_5. For $a\neq 1$ the contribution round the branch point tends to zero as $\varepsilon\to 0$.

Summing the real and imaginary parts we find for $a>1$

$$\text{Real:}\quad \frac{3}{2}\int_0^\infty \frac{\ln(t^3+1)}{t^3+a^3}\,dt + \pi\sqrt{3}\int_0^\infty \frac{dt}{t^3+a^3} - \pi\sqrt{3}\int_0^1 \frac{dt}{a^3-t^3} = \frac{\pi}{3a^2}(\pi+\sqrt{3}\,\ln(a^3-1)).$$

$$\text{Imag:}\quad \frac{-\sqrt{3}}{2}\int_0^\infty \frac{\ln(t^3+1)}{t^3+a^3}\,dt - \pi\int_0^\infty \frac{dt}{a^3-t^3} - \pi\int_0^1 \frac{dt}{t^3+a^3} = \frac{\pi}{3a^2}(\pi\sqrt{3}-\ln(a^3-1)).$$

If the imaginary part is multiplied by $\sqrt{3}$ and added to the real part, several terms cancel and leave

$$\int_0^\infty \frac{dt}{t^3+a^3} = \frac{2\pi\sqrt{3}}{9a^2},$$

a result which agrees with that in §6.6 for the case $a=1$. This is substituted back into either the real or imaginary part above. Using partial fractions

$$\int_0^1 \frac{1}{a^3-t^3}\,dt = \frac{-\pi\sqrt{3}}{18\,a^2} - \frac{1}{3\,a^2}\ln(a-1) + \frac{1}{6\,a^2}\ln\left(a^2+a+1\right) + \frac{\sqrt{3}}{3\,a^2}\tan^{-1}\left(\frac{a+2}{a\sqrt{3}}\right).$$

Collecting this together we conclude that

$$\int_0^\infty \frac{\ln(x^3+1)}{x^3+a^3}\,dx = \frac{\pi}{3\,a^2}\left[\sqrt{3}\,\ln\left(a^2+a+1\right) + 2\,\tan^{-1}\left(\frac{a+2}{a\,\sqrt{3}}\right) - \pi\right] \qquad \mathit{Eq.11.2a}$$

Though this was derived for $a>1$, the final expression has no singularities for any real positive value of a so there seems no reason to evaluate it at $a=1$ or for $a<1$. At $a=1$ it indeed reduces to

$$\int_0^\infty \frac{\ln(x^3+1)}{x^3+1}\,dx = \frac{\pi\ln 3}{\sqrt{3}} - \frac{\pi^2}{9} = 0\cdot 8960396 \qquad \mathit{Eq.11.2b}$$

in agreement with the value in Eq. 4.3 d of §4.7, quoted from Gradshteyn and Ryzhik's compendium. The integral for $a=2$ is

$$\frac{\pi}{12}\left(\sqrt{3}\,\ln 7 + 2\tan^{-1}\left(\tfrac{2}{\sqrt{3}}\right) - \pi\right) = 0\cdot 50866744$$

and for $a=\frac{1}{2}$ is

$$\frac{4\,\pi}{3}\left(\sqrt{3}\,\ln\left(\tfrac{7}{4}\right) + 2\,\tan^{-1}\left(\tfrac{5}{\sqrt{3}}\right) - \pi\right) = 1\cdot 26642473$$

Both these values agree with numerical quadrature.

Whilst it is gratifying to discover that a formula, derived under a limited assumption (here that $a > 1$) actually applies to many more cases, we should still give some thought as to *why* this happens. The case $a < 1$ can be understood as follows. The pole lies in the branch cut (Figure R) and so two semicircular indentations must be made around it, one in γ_4 and the other in γ_6. The integral round the whole loop is zero since there are no poles inside it, but in traversing these two indentations at $z = ae^{\imath\pi/3}$ one effectively evaluates $2\pi\imath$ times the residue at this point. Moreover, the contour is traversed in a clockwise direction around this point, so its contribution has the opposite sign to that which it would have if the pole were inside the contour, as when $a > 1$. This explains why the formula is the same whether $a > 1$ or $a < 1$, provided $a > 0$.

We now turn to the case $a = 1$. I proposed to evaluate this from first principles as a special case, in order to show how the singularities in the various constituent terms cancel. The sum of contributions from γ_1, γ_2 and γ_3 is

$$\frac{3}{2}\int_0^\infty \frac{\ln(t^3+1)}{t^3+1}\,dt + \frac{2\pi^2}{3} - \frac{\imath\sqrt{3}}{2}\int_0^\infty \frac{\ln(t^3+1)}{t^3+1}\,dt + \imath\frac{2\pi^2\sqrt{3}}{9}$$

where I have used the value for $\int_0^\infty dt/(t^3+1)$ from §6.6. Turning to γ_4, γ_5 and γ_6, we proceed as before and the sum of contributions from γ_4 and γ_6 is

$$\pi\,(-\sqrt{3}+\imath)\lim_{\varepsilon\to 0}\int_0^{1-\varepsilon}\frac{dt}{1-t^3}\quad .$$

and the integral evaluates as

$$\lim_{\varepsilon\to 0}\int_0^{1-\varepsilon}\frac{dt}{1-t^3} \;=\; \lim_{\varepsilon\to 0}\frac{1}{6}\left(\ln 3 - 2\,\ln\varepsilon + \frac{\pi}{\sqrt{3}}\right)\ .$$

On γ_5 $z = e^{\imath\pi/3} + \varepsilon\, e^{\imath\theta}$ so its contribution is

$$\lim_{\varepsilon\to 0}\int_{\frac{4\pi}{3}}^{-\frac{2\pi}{3}} \frac{\ln(3\varepsilon\, e^{\imath(2\pi/3+\theta)})}{3\varepsilon\, e^{\imath(2\pi/3+\theta)}}\,\imath\varepsilon\, e^{\imath\theta}\,d\theta$$

$$= \frac{1}{6}\lim_{\varepsilon\to 0}\int_{\frac{4\pi}{3}}^{-\frac{2\pi}{3}}\left[\sqrt{3}\,\ln(3\varepsilon) + \frac{2\pi}{3} + \theta\right]d\theta \;+\; \frac{\imath}{6}\int_{\frac{4\pi}{3}}^{-\frac{2\pi}{3}}\left[-\ln(3\varepsilon) + \frac{2\pi}{\sqrt{3}} + \sqrt{3}\,\theta\right]d\theta$$

The real part of the loop integral contributed by $\gamma_4 + \gamma_5 + \gamma_6$ is

$$\lim_{\varepsilon\to 0}\;\frac{\pi\sqrt{3}}{3}\ln\varepsilon - \frac{\pi\sqrt{3}}{6}\ln 3 - \frac{\pi^2}{6} \;-\; \frac{\pi\sqrt{3}}{3}\left[\ln\varepsilon + \ln 3\right] \;+\; \frac{\pi^2}{3}\ .$$

We see that the two terms in $\ln\varepsilon$ cancel precisely, meaning that in fact there is no singularity at $z = e^{\imath\pi/3}$. The remaining terms lead to the result for $\int_0^\infty \frac{\ln(t^3+1)}{t^3+1}\,dt$ given above.

11.3 Other integrals of similar structure involving $\ln(x^p+1)$, *p = 3, 4, 5, 6, etc.*

The contours of Figures 11.1 and 11.3 readily lend themselves to the evaluation of loop integrals of $z\ln(z^3+1)/(z^3 \pm a^3)$. The steps are in every respect equivalent to those set out in §10.2 and 10.3 for the cases of $\ln(z^3+1)/(z^3 \pm a^3)$, and the changes amount to little more that a few changes of sign and the incorporation of the extra z in the numerators of various terms. In passing one finds that

$$\int_0^\infty \frac{t}{t^3 - a^3}\, dt = \frac{\pi\sqrt{3}}{9a^2}\,,$$

and the results are as follows:

$$\int_0^\infty \frac{x\ln(x^3+1)}{x^3-a^3}\, dx = \frac{\pi}{3\,a}\left(\pi + \sqrt{3}\,\ln(a+1) + 2\,\tan^{-1}\left(\frac{a-2}{a\sqrt{3}}\right)\right).$$

$$\text{For a=1 :} \qquad \int_0^\infty \frac{x\ln(x^3+1)}{x^3-1}\, dx = \frac{2\pi^2}{9} + \frac{\pi\ln 2}{\sqrt{3}} = 3\cdot 4504753$$

$$\text{and for a=2 :} \qquad \int_0^\infty \frac{x\ln(x^3+1)}{x^3-8}\, dx = \frac{\pi^2}{6} + \frac{\pi\ln 3}{2\sqrt{3}} = 2\cdot 641265$$

These must be regarded as the Cauchy principal value of the integral through the singularity at $x = a$. As a ready variation of §10.3

$$\int_0^\infty \frac{x\ln(x^3+1)}{x^3+a^3}\, dx = \frac{\pi}{3\,a}\left[\sqrt{3}\,\ln\left(a^2+a+1\right) - 2\,\tan^{-1}\left(\frac{a+2}{a\,\sqrt{3}}\right) + \pi\right]$$

At $a = 1$ it reduces to

$$\int_0^\infty \frac{x\ln(x^3+1)}{x^3+1}\, dx = \frac{\pi\ln 3}{\sqrt{3}} + \frac{\pi^2}{9} = 3\cdot 089285 \qquad Eq.11.3$$

in agreement with the value in Eq. 4.3 e of §4.7, quoted from Gradshteyn and Ryzhik's tome. The integral for $a = 2$ is

$$\frac{\pi}{6}\left(\sqrt{3}\,\ln 7 \;-\; 2\tan^{-1}\left(\tfrac{2}{\sqrt{3}}\right) + \pi\right) = 2\cdot 51215574.$$

In principle the contours of Figures 11.1 and 11.3 can be further adapted to values of exponent p other than 3 simply by adjusting the angle of the contour wedge and its branch cut to be at $2\pi/p$ and π/p respectively. I consider the case

$$I_p = \int_0^\infty \frac{\ln(x^p+1)}{x^p+a^p}\, dx \;\text{ by integrating }\; \oint \frac{\ln(z^p+1)}{z^p+a^p}\, dz \text{ around Figure 11.3.}$$

The contribution from $\gamma_1 + \gamma_2 + \gamma_3$ is

$$(1-c_2)I_p \;+\; 2\pi s_2 Y_p \;-\; \imath\, s_2\, I_p \;-\; \imath\, 2\pi c_2 Y_p$$

where I_p is the required real integral and

$$Y_p \;=\; \int_0^\infty \frac{dt}{t^p+a^p} = \frac{\pi}{p a^{p-1}}\left[\frac{2\sin(\frac{\pi}{p})}{1-\cos(\frac{2\pi}{p})}\right] \qquad \text{for } p \text{ a positive integer} \geq 2\,,$$

the evaluation coming from §6.9. In these $s_1 = \sin(\pi/p)$, $c_1 = \cos(\pi/p)$, $s_2 = \sin(2\pi/p)$, $c_2 = \cos(2\pi/p)$.

The contribution from $\gamma_4 + \gamma_5 + \gamma_6$ is

$$2\pi(-s_1 + \imath\, c_1)\, Z_p \quad \text{where} \quad Z_p = \int_0^1 \frac{dt}{a^p - t^p}.$$

Also $2\pi\imath\times$(the residue at $a\, e^{\imath\pi/p}$) is

$$\frac{2\pi}{p\, a^{p-1}}\left[s_1 \ln(a^p - 1) + c_1\pi + \imath s_1 \pi - \imath\, c_1 \ln(a^p - 1)\right].$$

Adding the real and imaginary parts of the loop integral, we find

$$\text{Real:} \qquad (1 - c_2) I_p \;+\; 2\pi\, s_2 Y_p \;-\; 2\pi s_1 Z_p = \frac{2\pi}{p\, a^{p-1}}\left[s_1 \ln(a^p - 1) + c_1 \pi\right]$$

$$\text{Imag:} \qquad -\, s_2\, I_p \;-\; 2\pi c_2 Y_p \;+\; 2\pi c_1\, Z_p = \frac{2\pi}{p\, a^{p-1}}\left[s_1 \pi - c_1 \ln(a^p - 1)\right].$$

These are a pair of simultaneous linear equations which, when combined as in §10.3, give the value of Y_p already found in §6.9 and quoted above. This indeed in an alternative derivation of the value of the integral Y_p. The equations by themselves cannot furnish values for both Z_p and I_p, so there seems no alternative to evaluating Z_p directly. This can be done by splitting $1/(a^p - t^p)$ into partial fractions in terms of the complex roots of $a^p - t^p = 0$. Thus

$$\frac{1}{t^p - a^p} = \frac{1}{t - a} + \sum_{k=1}^{p-1} \frac{\omega_{2k}}{t - a\,\omega_{2k}} \qquad \text{where } \omega_{2k} = e^{\imath 2\pi k/p}.$$

This can now be integrated as a sum of complex logarithms. Unfortunately, for most values of p the results are complicated. The first few cases are

$$Z_4 = \frac{1}{4\, a^3}\left[\ln(a+1) - \ln(a-1) + 2\,\tan^{-1}\left(\frac{1}{a}\right)\right] \quad \text{which leads to}$$

$$\int_0^\infty \frac{\ln(x^4+1)}{x^4 + a^4}\, dx = \frac{\pi\sqrt{2}}{4\, a^3}\left[\ln\left(a^2+1\right) + 2\,\ln(a+1) + 2\,\tan^{-1}\left(\frac{1}{a}\right) - \pi\right]$$

When $a = 1$ this reduces to

$$\int_0^\infty \frac{\ln(x^4+1)}{x^4 + 1}\, dx = \frac{\pi\sqrt{2}}{8}\,(6\,\ln 2 - \pi) = 0\cdot 5649628,$$

a value confirmed by numerical quadrature. For $p = 5$ the complicated Z_5 is equal to

$$\frac{1}{20a^4}\left[\left(\sqrt{5}+1\right)\ln\left(\frac{2\,a^2 + \left(\sqrt{5}+1\right)\,a + 2}{2}\right) + \left(1 - \sqrt{5}\right)\ln\left(\frac{2\,a^2 + \left(1 - \sqrt{5}\right)\,a + 2}{2}\right) - 4\ln(a-1)\right.$$

$$\left. + 2\sqrt{2}\sqrt{\sqrt{5}+5}\tan^{-1}\left(\frac{\sqrt{2}\,\left(\sqrt{5}+5\right)^{\frac{3}{2}}}{4\,\left(\sqrt{5}\,a + 5\,a - \sqrt{5}\right)}\right) + 2\sqrt{2}\sqrt{5 - \sqrt{5}}\tan^{-1}\left(\frac{\sqrt{2}\sqrt{5-\sqrt{5}}\,\left(\sqrt{5}-5\right)}{4\,\left(\sqrt{5}\,a - 5\,a - \sqrt{5}\right)}\right)\right]$$

Solving for the integral for the case $a = 1$ we obtain the awkward but exact formula

$$I_5 = \int_0^\infty \frac{\ln(x^5+1)}{x^5+1}\,dx = \frac{-\pi}{10\,(c_2-1)^2}\Bigg[8\,\pi\,s_1 s_2 + 2s_1\,(c_2-1)\sqrt{2}\sqrt{\sqrt{5}+5}\,\tan^{-1}\left(\frac{(\sqrt{5}+5)^{\frac{3}{2}}}{10\sqrt{2}}\right)$$

$$+s_1\,(c_2-1)\sqrt{5}\,\ln\left(\frac{5+\sqrt{5}}{5-\sqrt{5}}\right) + 2s_1\,(1-c_2)\sqrt{2}\sqrt{5-\sqrt{5}}\,\tan^{-1}\left(\frac{\sqrt{2}\sqrt{5-\sqrt{5}}\,(\sqrt{5}-5)}{20}\right)$$

$$+s_1\,(c_2-1)\ln 3125 + 4\,\pi\,c_1(c_2-1)\Bigg]$$

where $s_1 = \sin(\pi/5)$, etc. as above. The value of this is $0 \cdot 41453484$.
Finally, for $p = 6$ the general solution is

$$\int_0^\infty \frac{\ln(x^6+1)}{x^6+a^6}\,dx = \frac{\pi}{6\,a^5}\left[3\,\ln\left(a^3+2\,a^2+2\,a+1\right)+\ln\left(a^3+1\right)+2\sqrt{3}\,\left(\tan^{-1}\left(\tfrac{\sqrt{3}\,a}{a^2-1}\right)-\pi\right)\right]$$

which reduces for $a = 1$ to

$$\int_0^\infty \frac{\ln(x^6+1)}{x^6+1}\,dx = \frac{\pi}{6}(4\ln 2 + 3\ln 3 - \pi\sqrt{3}) = 0 \cdot 3283108.$$

12. Attempt to analyse and evaluate $\int_0^1 \frac{\ln(x^3+1)}{x^2+1}\,dx$ using series

I have set myself the challenge of getting as far as I possibly can towards a closed form evaluation of this integral. I confess now that I have been unable to find a wholly closed form expression for it; a small part stubbornly remains as an infinite series. Indeed I suspect that a closed form does not exist. So let me make this conjecture;

> "The only integrals of the forms
>
> $$\int_0^1 \frac{\ln(x^p+1)}{x^2+1}\,dx \quad \text{and} \quad \int_0^\infty \frac{\ln(x^p+1)}{x^2+1}\,dx$$
>
> which have a closed form evaluation are those with $p = 1$ or 2, or p an integer multiple of 4, or p twice an odd prime. In particular no closed form evaluation exists if p is a fraction, or an odd prime, or an odd multiple of an odd prime."

I herewith invite other enthusiasts to prove this either correct or incorrect.

In §12.1 I explain why I think that evaluation by complex contour integration is intractable. Next, in §12.2, I present my attempt to evaluate it using a series expansion. In general, determining an integral from an infinite series is usually the last resort of the desperate; it is more usual to work the opposite way and use an integral to evaluate an infinite series. I have failed to find a closed form summation of the series expansion of the integral. However, I have found some interesting sub-series, which may throw some limited light on the structure of the integral.

12.1 Why most real integrals involving $\ln(x^p + 1)$ *for* $p = 3$ *and other odd primes are intractable*

In §11 I showed how to solve in closed form $\int_0^1 \ln(x^3+1)/(x^3+1)\,dx$. This is possible by contour integration because the argument of the logarithm is the same as the denominator – or, to be more specific, has the same zeros as the denominator. However my attempts to find a contour to evaluate $\int_0^1 \ln(x^3+1)/x^2+1$ have all failed. I have been reduced to attempting integration using infinite series, as described in full in §12.2. So why is the latter integral so intractable while the first can be done?

In attempting a contour integration we are free to choose

a) the argument of the logarithm, with a view to generating the desired real integral,
b) the multiplying rational function, also with a view to generating the desired real integral,
c) the contour, avoiding poles and branch cuts.

Regarding a) we saw in §9 that the real integral we want does not have to feature directly on the contour, but may be generated by some related though non-obvious complex function. For instance, in §9 we determined that

$$\int_0^\infty \frac{\ln(x^4+1)}{x^2+1}\,dx = \pi\ln(2+\sqrt{2}) \text{ by evaluating } \oint \frac{\ln(z+e^{\imath\pi/4})}{z^2+1}\,dz$$

However my attempts to find an argument which will generate $\ln(x^3+1)$ *and* provide x^2+1 in the denominator have all failed. Regarding b), the 'limiting wedge contour' discussed in §10.2 manipulates the rational function but still fails to yield the required integral. Regarding c), the methods used in §11 to evaluate

$$\text{i)} \int_0^\infty \frac{\ln(x^3+1)}{x^3+1}\,dx \text{and ii)} \int_0^\infty \frac{\ln(x^{2n}+1}{x^2+1}\,dx$$

give some insight into why it is probably not possible to use contour integration to evaluate $\int_0^\infty \frac{\ln(x^3+1)}{x^2+1}\,dx$. Consider in integral i) the $2\pi/3$ sector contour. Any segment of contour which gives real values for z^3+1 must give complex values for z^2+1 in the denominator. Then, when the denominator is rationalised at a later step in the calculation, it is multiplied by its complex conjugate. This both creates unwanted square terms in the denominator and creates additional intractable integrals involving arctangents. Consider in case b) the device of having a complex argument for the logarithm while selecting sections of contour on which z^2+1 is real. When the logarithm is evaluated we find its square modulus and its argument, and this process creates square terms and additional intractable integrals involving arctangents. I have been unable to find a contour for our required integral which avoids these problems.

12.2 Series expansion of $\dfrac{\ln(x^3+1)}{x^2+1}\,dx$

Notwithstanding the expectation that integration using expansions in infinite series is unlikely to succeed, I will get as far with this approach as I can, hoping that it sheds some insight into the structure of this integral.

Numerical quadrature gives the value of the integral from 0 to 1 as 0·12470402235, and of the integral from 0 to infinity as 2·99730482723. I showed in §9.6 the relation between these, why $2\times 0{\cdot}12470402235 + 3\times\mathbf{G} = 2\cdot 99730482723$. The binomial expansion of $1/(x^2+1)$ is strictly valid only for $|x| < 1$, but I will attempt to evaluate the integral $\int_0^1$ using this series, taking the limit $x \to 1$. This gives the infinite series

$$\int_0^1 \frac{\ln(x^3+1)}{x^2+1}\,dx = \lim_{\varepsilon\to 0}\int_0^{1-\varepsilon} \frac{\ln(x^3+1)}{x^2+1}\,dx = I_0 - I_2 + I_4 - I_6 + \cdots \qquad Eq12.1$$

$$\text{where } I_n = \lim_{c\to 1}\int_0^c x^n \ln(x^3+1)\,dx.$$

The function I_n has a recursion relation which is obtained by integration by parts using $u = x^{n-2}$, $dv = x^2\log(x^3+1)\,dx$:

$$I_{n+3} = \frac{1}{n+4}\left[-(n+1)I_n + 2\ln 2 - \frac{3}{n+4}\right] \qquad Eq12.2$$

Since we are stepping in Eq. 12.1 in steps of 2 but the recursion relation steps in 3, we have to advance in steps which are the lowest common multiple of these, which is 6. A double application of Eq. 12.2 gives

$$I_{n+6} = \frac{1}{n+7}\left[(n+1)I_n + 3\left(\frac{1}{n+4} - \frac{1}{n+7}\right)\right] \qquad Eq12.3$$

With this we construct the series in Eq 12.1 as three the interlocking series, which I will name the I_0, I_2 and I_4 series :

$$\begin{aligned}
&I_0 \text{ series : } I_0 - I_6 + I_{12} - I_{18} + I_{24} - \cdots \\
&I_2 \text{ series : } I_2 - I_8 + I_{14} - I_{20} + I_{26} - \cdots (\text{ use negative of this in sum}) \\
&I_4 \text{ series : } I_4 - I_{10} + I_{16} - I_{22} + I_{28} - \cdots
\end{aligned}$$

Taking the limit $\varepsilon \to 0$ the base cases in the recursion relation are

$$\begin{aligned}
I_0 &= \frac{\pi}{\sqrt{3}} + 2\ln 2 - 3 \\
3I_2 &= 2\ln 2 - 1 \\
5I_4 &= \frac{-\pi}{\sqrt{3}} + 2\ln 2 + \frac{9}{10}
\end{aligned}.$$

(The reason for the multiples of 3 and 5 will become clear shortly.) Now use the 6-stepping recursion relation, Eq 12.3, to develop these three series:

$$\begin{aligned}
+\text{ sign : } & I_0 = \tfrac{1}{1} I_0 \\
-\text{ sign : } & I_6 = \tfrac{1}{7}\left[I_0 + 3\left(\tfrac{1}{4} - \tfrac{1}{7}\right)\right] \\
+\text{ sign : } & I_{12} = \tfrac{1}{13}\left[I_0 + 3\left(\tfrac{1}{4} - \tfrac{1}{7} + \tfrac{1}{10} - \tfrac{1}{13}\right)\right] \\
-\text{ sign : } & I_{18} = \tfrac{1}{19}\left[I_0 + 3\left(\tfrac{1}{4} - \tfrac{1}{7} + \tfrac{1}{10} - \tfrac{1}{13} + \tfrac{1}{16} - \tfrac{1}{19}\right)\right] \\
+\text{ sign : } & I_{24} = \tfrac{1}{23}\left[I_0 + 3\left(\tfrac{1}{4} - \tfrac{1}{7} + \tfrac{1}{10} - \tfrac{1}{13} + \tfrac{1}{16} - \tfrac{1}{19} + \tfrac{1}{22} - \tfrac{1}{25}\right)\right] \\
-\text{ sign : } & I_{30} = \cdots
\end{aligned}$$

The + or − sign in the first column above indicates whether the corresponding I_n term is added or subtracted in summing the series.

$$\begin{aligned}
-\text{ sign : } & I_2 = \tfrac{1}{3} 3I_2 \\
+\text{ sign : } & I_8 = \tfrac{1}{9}\left[3I_2 + 3\left(\tfrac{1}{6} - \tfrac{1}{9}\right)\right] \\
-\text{ sign : } & I_{14} = \tfrac{1}{15}\left[3I_2 + 3\left(\tfrac{1}{6} - \tfrac{1}{9} + \tfrac{1}{12} - \tfrac{1}{15}\right)\right] \\
+\text{ sign : } & I_{20} = \tfrac{1}{21}\left[3I_2 + 3\left(\tfrac{1}{6} - \tfrac{1}{9} + \tfrac{1}{12} - \tfrac{1}{15} + \tfrac{1}{18} - \tfrac{1}{21}\right)\right] \\
-\text{ sign : } & I_{26} = \tfrac{1}{27}\left[3I_2 + 3\left(\tfrac{1}{6} - \tfrac{1}{9} + \tfrac{1}{12} - \tfrac{1}{15} + \tfrac{1}{18} - \tfrac{1}{21} + \tfrac{1}{24} - \tfrac{1}{27}\right)\right] \\
+\text{ sign : } & I_{32} = \cdots
\end{aligned}$$

and

$$\begin{aligned}
+\text{ sign}: \quad & I_4 = \tfrac{1}{5}\, 5I_0 \\
-\text{ sign}: \quad & I_{10} = \tfrac{1}{11}\left[5I_4 + 3\left(\tfrac{1}{8} - \tfrac{1}{11}\right)\right] \\
+\text{ sign}: \quad & I_{16} = \tfrac{1}{17}\left[5I_4 + 3\left(\tfrac{1}{8} - \tfrac{1}{11} + \tfrac{1}{14} - \tfrac{1}{17}\right)\right] \\
-\text{ sign}: \quad & I_{22} = \tfrac{1}{23}\left[5I_4 + 3\left(\tfrac{1}{8} - \tfrac{1}{11} + \tfrac{1}{14} - \tfrac{1}{17} + \tfrac{1}{20} - \tfrac{1}{23}\right)\right] \\
+\text{ sign}: \quad & I_{28} = \tfrac{1}{25}\left[5I_4 + 3\left(\tfrac{1}{8} - \tfrac{1}{11} + \tfrac{1}{14} - \tfrac{1}{17} + \tfrac{1}{20} - \tfrac{1}{23} + \tfrac{1}{26} - \tfrac{1}{29}\right)\right] \\
-\text{ sign}: \quad & I_{34} = \cdots
\end{aligned}$$

In each I_n there is a term in either I_0, $3I_2$ or $5I_4$. Separate these out and add them together to form a partial sum; call it $\mathbf{J}$. The remaining terms, involving $3\times$(finite sum of reciprocal integers) are added to form $3\mathbf{K}$. Thus

$$\int_0^1 \frac{\ln(x^3+1)}{x^2+1}\, dx = \mathbf{J} + 3\mathbf{K}$$

where

$$\mathbf{J} = I_0\left[\tfrac{1}{1} - \tfrac{1}{7} + \tfrac{1}{13} - \tfrac{1}{19} + \cdots\right] - 3I_2\left[\tfrac{1}{3} - \tfrac{1}{9} + \tfrac{1}{15} - \tfrac{1}{21} + \cdots\right] + 5I_4\left[\tfrac{1}{5} - \tfrac{1}{11} + \tfrac{1}{17} - \tfrac{1}{23} + \cdots\right]$$

and

$$\begin{aligned}
\mathbf{K} =& \left(\tfrac{1}{4} - \tfrac{1}{7}\right)\left[-\tfrac{1}{7} + \tfrac{1}{13} - \tfrac{1}{19} + \tfrac{1}{25} - \cdots\right] + \left(\tfrac{1}{10} - \tfrac{1}{13}\right)\left[\tfrac{1}{13} - \tfrac{1}{19} + \tfrac{1}{25} - \cdots\right) \\
& + \left(\tfrac{1}{16} - \tfrac{1}{19}\right)\left(-\tfrac{1}{19} + \tfrac{1}{25} - \tfrac{1}{31} + \cdots\right] + \left(\tfrac{1}{22} - \tfrac{1}{25}\right)\left[\tfrac{1}{25} - \tfrac{1}{31} + \cdots\right] \\
& + \cdots \\
& \left(\tfrac{1}{6} - \tfrac{1}{9}\right)\left[\tfrac{1}{9} - \tfrac{1}{15} + \tfrac{1}{21} - \tfrac{1}{27} + \cdots\right] + \left(\tfrac{1}{12} - \tfrac{1}{15}\right)\left[-\tfrac{1}{15} + \tfrac{1}{21} - \tfrac{1}{27} - \cdots\right] \\
& + \left(\tfrac{1}{18} - \tfrac{1}{21}\right)\left[\tfrac{1}{21} - \tfrac{1}{27} + \tfrac{1}{33} + \cdots\right] \\
& + \cdots \\
& \left(\tfrac{1}{8} - \tfrac{1}{11}\right)\left[-\tfrac{1}{11} + \tfrac{1}{17} - \tfrac{1}{23} + \tfrac{1}{29} - \cdots\right] + \left(\tfrac{1}{14} - \tfrac{1}{17}\right)\left[\tfrac{1}{17} - \tfrac{1}{23} + \tfrac{1}{29} - \cdots\right] \\
& + \left(\tfrac{1}{20} - \tfrac{1}{23}\right)\left[-\tfrac{1}{23} + \tfrac{1}{29} - \tfrac{1}{35} + \cdots\right] \\
& + \cdots
\end{aligned}$$

I will deal with these parts $\mathbf{J}$ and $\mathbf{K}$ in turn.

First $\mathbf{J}$. Fortunately this can be summed in closed form by using the integral forms of the series

$$\tfrac{1}{1} - \tfrac{1}{7} + \tfrac{1}{13} - \tfrac{1}{19} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k}{6k+1} = \int_0^1 \frac{1}{x^6+1}\, dx = \tfrac{1}{6}[\pi + \ln(2+\sqrt{3})] = 0 \cdot 903772$$

$$\tfrac{1}{3} - \tfrac{1}{9} + \tfrac{1}{15} - \tfrac{1}{21} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k}{6k+3} = \int_0^1 \frac{x^2}{x^6+1}\, dx = \tfrac{\pi}{12} = 0 \cdot 261799$$

$$\tfrac{1}{5} - \tfrac{1}{11} + \tfrac{1}{17} - \tfrac{1}{23} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k}{6k+5} = \int_0^1 \frac{x^4}{x^6+1}\, dx = \tfrac{1}{6}[\pi - \ln(2+\sqrt{3})] = 0 \cdot 143426\ .$$

Combining these with the values of I_0, I_2 and I_4 gives exactly

$$\mathbf{J} = \left(\frac{\pi}{3} - \frac{13}{20}\sqrt{3}\right)\ln(2+\sqrt{3}) \;+\; \frac{\pi}{2}\ln 2 \;-\; \frac{4\pi}{15} = 0 \cdot 1474753961. \qquad Eq.12.4$$

Taken with the numerical quadrature value of the integral $(0 \cdot 12470402235)$, this value of **J** leads us to expect that **K** must be $-0 \cdot 0075904579$. So let us turn to the series which make up **K** and see if this holds true.

12.3 Structure and evaluation of the series **K**

I have been unable to sum exactly the series which make up **K**. (See footnote.) This section discussed three approaches to determining its value and revealing its constitution.

A diagram of the values is helpful so I have drawn up the following matrix to represent the denominators of the various terms contributing to **K**. Each term has two factors in the denominator and these are denoted by the row and column. The sign in the cell shows whether it is added or subtracted. As an example, the + sign in cell of row 9, column 6 means that a term $+1/(6\times9)$ appears in **K**.

Table of denominators of terms in **K**

	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
7	−			+																		
8																						
9			+			−																
10																						
11					−			+														
12																						
13	+			−			+			−												
14																						
15			−			+			−			+										
16																						
17					+			−			+			−								
18																						
19	−			+			−			+			−			+						
20																						
21			+			−			+			−			+			−				
22																						
23					−			+			−			+			−			+		
24																						
25	+			−			+			−			+			−			+			−
26	···																					

Note that there are no entries in column 5.

The first approach to evaluating this is direct numerical addition. The terms down to row labelled 20,001 add to about $-0 \cdot 007590$. This gives confidence the analysis so far. However direct addition shows that the series converges slowly and high accuracy is not readily attained.

Here is a free English language lesson. Note that I am using correct English grammar here by saying 'to sum exactly'. Much current usage, even by BBC news readers, would say 'to exactly sum', but this is crude and incorrect in traditional British English grammar. The reason is that the infinitive of the verb is 'to sum'and this should not be split by any qualifying adverb such as 'exactly'. In the past, exception was made only for emphasis, when the adverb 'exactly'had much greater significance than usual. In most languages the infinitive of the verb is a single word so the option of splitting it does not arise (*e.g.* ajouter, additionner in French). For English, the rot set in when Captain Kirk of the *Star Ship Enterprise* decided 'to boldly go'. Now back to the maths.

The main diagonal in the $\searrow$ direction, bounding the triangular array, represents the series

$$\tfrac{1}{7^2} - \tfrac{1}{9^2} + \tfrac{1}{11^2} - \tfrac{1}{13^2} + \cdots = 1 - \tfrac{1}{3^2} + \tfrac{1}{5^2} - \mathbf{G} = 0 \cdot 0129232947$$

where $\mathbf{G}$ is Catalan's constant (see §5.1). Note also that the terms along any perpendicular diagonal (top right to bottom left, $\swarrow$) are all much the same size. For instance, along the diagonal containing 9^2, $9 \times 9 = 81$, $8 \times 11 = 88$, $7 \times 13 = 91$, $6 \times 15 = 90$ and $4 \times 19 = 76$. This gives us our next method for estimating $\mathbf{K}$. In the $\swarrow$ diagonal containing n^2 there are $n - 4$ entries (because the columns in 1, 2, 3 and 5 are blank). The contribution from this one diagonal therefore amounts to about $(n-4)/n^2$ in absolute value. Since only odd values of n appear in the table,

$$\mathbf{K} \approx \sum_{k=2}^{\infty} \frac{(2k+1)-4}{(2k+1)^2} = \left[-\tfrac{\pi}{4} + 1 - \tfrac{1}{3}\right] - 4\left[-\mathbf{G} + 1 - \tfrac{1}{3^2}\right]$$

where I have used the Gregory series for $\tan^{-1} 1 = \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$. This works out to be $4\mathbf{G} - \frac{\pi}{4} - \frac{26}{9} = -0 \cdot 010425$. Though this is not an accurate approximation it does give some insight because a) it is of the correct order of magnitude, b) it indicates the presence of both π and Catalan's constant in $\mathbf{K}$. More accurate approximations can be obtained by summing the first few diagonals exactly. For example, the first four $\swarrow$ diagonals, down to and including the one in 11^2, sum to $0 \cdot 01965$ whereas the approximation used for these rows is $0 \cdot 01735$. Use of the correct sum for these four diagonals changes the estimate of $\mathbf{K}$ to $-0 \cdot 00812$ which is much closer to $-0 \cdot 00759$. Regrettable, the difference between the exact sum and the approximation appears not to decrease indefinitely as n increases. In fact two sub-series can be separated, characterised by the sign of the last term: one for $n = 1 \bmod 4$, for which the error converges from above to be about $-0 \cdot 0022$ at $n = 1001$, and the other for $n = 3 \bmod 4$, for which the error converges from below to about $-0 \cdot 0034$. It might be interesting to give this further investigation, but not here!

My third and most accurate approach to summing the series for $\mathbf{K}$ is to sum in the other diagonal direction, $\searrow$, parallel to the boundary diagonal. We have already noted that the main diagonal sums to $1 - \frac{1}{3^2} + \frac{1}{5^2} - \mathbf{G} =$. Call this $Diag_0$. In the first parallel diagonal, $Diag_1$, the two factors in the denominator differ by 3 *e.g.* 4 and 7, 6 and 9, 8 and 11. To sum the parallel diagonals, $Diag_1$, $Diag_2$ etc., split each denominator into two using the identity

$$\frac{1}{n(n+k)} = \frac{1}{k}\left[\frac{1}{n} - \frac{1}{n+k}\right] \qquad Eq12.5$$

This splitting replaces $Diag_1$ by

$$\tfrac{1}{3}\left[-\tfrac{1}{4} + \tfrac{1}{7} + \tfrac{1}{6} - \tfrac{1}{9} - \tfrac{1}{8} + \tfrac{1}{11} + \tfrac{1}{10} - \tfrac{1}{13} - \cdots\right]$$

which can be reorganised as

$$\tfrac{1}{3}\left[-\tfrac{1}{4} + \tfrac{1}{6} - \tfrac{1}{8} + \tfrac{1}{10} - \cdots \quad + \tfrac{1}{7} - \tfrac{1}{9} + \tfrac{1}{11} - \tfrac{1}{13} + \cdots\right].$$

Now introduce the notation

$$S_n = \tfrac{1}{n} - \tfrac{1}{n+2} + \tfrac{1}{n+4} - \tfrac{1}{n+6} + \cdots.$$

The sum of each $\swarrow$ diagonal can now be written as follows:

$$\begin{array}{ll}
Diag_0: & 1-\frac{1}{3^2}+\frac{1}{5^2}-\mathbf{G} \\
Diag_1: & \frac{1}{3}\left[-S_4+S_7\right] \\
Diag_2: & \frac{1}{6}\left[-S_7+S_{13}\right] \\
Diag_3: & \frac{1}{9}\left[+S_4-S_{13}\right] \\
Diag_4: & \frac{1}{12}\left[+S_7-S_{19}\right] \\
Diag_5: & \frac{1}{15}\left[-S_5+S_{19}\right] \\
Diag_6: & \frac{1}{18}\left[-S_7+S_{25}\right] \\
Diag_7: & \frac{1}{21}\left[+S_4-S_{25}\right] \\
Diag_8: & \frac{1}{24}\left[+S_7-S_{31}\right] \\
Diag_9: & \cdots
\end{array}$$

Note the patterns in signs and indices of S_n. For each diagonal the first term has the repeated pattern unit $-S_4$, $-S_7$, $+S_4$, $+S_7$, whilst the second term follows $+S_7$, $+S_{13}$, $-S_{13}$, $-S_{19}$, $+S_{19}$, $+S_{25}$, $-S_{25}$, $-S_{31}$, etc. Adding all these gives

$$\mathbf{K}=1-\frac{1}{3^2}+\frac{1}{5^2}-\mathbf{G}+S_4\left[-\frac{1}{3}+\frac{1}{9}-\frac{1}{15}+\frac{1}{21}-\cdots\right]+S_7\left[\left(\frac{1}{3}\right)-\frac{1}{6}+\frac{1}{12}-\frac{1}{18}+\frac{1}{24}-\cdots\right]$$

$$+S_{13}\left[\frac{1}{6}-\frac{1}{9}\right]-S_{19}\left[\frac{1}{12}-\frac{1}{15}\right]+S_{25}\left[\frac{1}{18}-\frac{1}{21}\right]-S_{31}\left[\frac{1}{24}-\frac{1}{27}\right]+\cdots.$$

We met the series multiplying S_4 in the evaluation of $\mathbf{J}$: it is $-\pi/12$. The series multiplying S_7 is

$$\frac{1}{3}-\frac{1}{6}\left[1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots\right]=\frac{1}{3}-\frac{1}{6}\ln 2.$$

S_4 itself is $-\frac{1}{2}\left[\left(-1+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}+\cdots\right)+1\right]=\frac{1}{2}(1-\ln 2)$, while $S_7=1-\frac{1}{3}+\frac{1}{5}-\frac{\pi}{4}$. Moreover, Eq. 12.5 can be used in reverse to recombine the partial fractions multiplying S_{13} *et. seq.*. Also we can take out a factor of 6 from the S_{13} and higher terms. Taken together these substitutions give

$$\mathbf{K}=-\mathbf{G}+\left(\frac{\pi}{12}-\frac{13}{90}\right)\ln 2-\frac{\pi}{8}+\frac{274}{225}+\frac{1}{3}\left[\frac{S_{13}}{2.3}-\frac{S_{19}}{4.5}+\frac{S_{25}}{6.7}-\cdots\right] \qquad Eq.12.6$$

$$\text{or} \quad 3\mathbf{K}=-0\cdot 02862794996+\mathbf{S} \qquad \text{where } \mathbf{S}=\frac{S_{13}}{2.3}-\frac{S_{19}}{4.5}+\frac{S_{25}}{6.7}-\cdots$$

So far this is exact. So evaluation of the required integral comes down to the evaluation of the infinite series $\mathbf{S}$. I cannot see how this can be done in closed form; hence my conjecture that a closed form expression does no exist. However we can make numerical calculations of the constituent S_n to a fair accuracy as follows.

Each S_n consists of the series for $\tan^{-1} 1$ with all terms before $1/n$ deleted. For the smaller values of n these can be readily computed exactly, and indeed there seems no alternative to so doing. For instance,

$$S_{13}=\frac{\pi}{4}-\frac{2578}{3465} \quad \text{and} \quad -S_{19}=\frac{\pi}{4}-\frac{622637}{765765}=S_{13}-\frac{1}{13}+\frac{1}{15}-\frac{1}{17}.$$

Note the alternating sign. It means that there are effectively two sub-series, depending on whether the last deleted term, $1/(n-2)$, was $+$ or $-$ in sign,. This implies significant cancelling between adjacent terms and suggests that it may be useful to take adjacent terms in pairs. To give some idea of the magnitude of adjacent pairs,

$$
\begin{array}{ll}
S_{13} = 0 \cdot 04138661939 & \\
S_{19} = 0 \cdot 02769332028 & S_{13}/(2 \times 3) - S_{19}/(4 \times 5) = 0 \cdot 00551310388 \\
S_{25} = 0 \cdot 02079747192 & \\
S_{31} = 0 \cdot 01664824967 & S_{25}/(6 \times 7) - S_{31}/(8 \times 9) = 0 \cdot 00026395221 \\
S_{37} = 0 \cdot 01387821312 & \\
S_{43} = 0 \cdot 01189803217 & S_{37}/(10 \times 11) - S_{43}/(12 \times 13) = 4 \cdot 98961367866 \times 10^{-5} \\
S_{49} = 0 \cdot 01041215530 & \\
S_{55} = 0 \cdot 00925608935 & S_{49}/(14 \times 15) - S_{55}/(16 \times 17) = 1 \cdot 555195166 \times 10^{-5} \\
S_{61} = 0 \cdot 00833102172 & \\
S_{67} = 0 \cdot 00757402041 & S_{61}/(18 \times 19) - S_{67}/(20 \times 21) = 6 \cdot 3263307066 \times 10^{-6} \\
S_{73} = 0 \cdot 00694310614 & \\
S_{79} = 0 \cdot 00640920365 & S_{73}/(22 \times 23) - S_{79}/(24 \times 25) = 3 \cdot 0395475576 \times 10^{-6}
\end{array}
$$

The sum of these terms from $S_{13}/(2 \times 3)$ up to and including $-S_{79}/(24 \times 25)$ is $0 \cdot 071813125114\,\pi - 0 \cdot 21975571623 = 0 \cdot 005851870063$ which, given exactly, is

$$\frac{7690078169}{107084577600}\,\pi - \frac{3773972052459434382697007162701696478947l}{17173487530841892974252090105268045321000} .$$

Though this is a many digit fraction, it has a simple structure and shows the continuing role of π. Using the Maxima program I have summed this series for $\mathbf{S}$ directly to 250 and then 400 terms. The value obtained is $\mathbf{S} = 0 \cdot 0058565761$.

12.4 Further analysis and asymptotic behaviour of the series for **S**

Rather than just accept this numerical computation of the series for **S**, I have looked into its structure and in particular its asymptotic behaviour. The next couple of pages explore this. **S** can be written as

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1} S_{6k+7}}{2k(2k+1)}$$

and, because of the alternating signs and consequent cancellation, it seems appropriate to the sum adjacent terms for k and $k+1$. We obtain the cumbersome expression

$$\frac{S_{6k+7}}{2k(2k+1)} + \frac{1}{(2k+2)(2k+3)} \left[S_{6k+7} - \frac{1}{6k+7} + \frac{1}{6k+9} - \frac{1}{6k+11} \right] .$$

which is always positive in sign. For large k this sum of pairs can be expanded as a Taylor series in $1/k$ as

$$\frac{S_{6k+7}}{2\,k^2} - \frac{18\,S_{6k+7} + 1}{24\,k^3} + \cdots ,$$

showing that successive terms tend to zero faster than $1/k^2$ (faster because the S_{6k+7} also decrease).

In order to sum $\mathbf{S}$ we now need a reliable approximation for S_n, such as an asymptotic series, as n tends to infinity. I have considered the Euler-Maclaurin sum formula to find such a series. This formula effectively approximates a finite sum by a finite integral plus a series, and an error term whose magnitude can be estimated. I believe I have found an even more accurate series, similar in structure to that obtained from the Euler-Maclaurin method. It is derived by considering that $\tan^{-1} N$ is a reasonable approximation to the sum over $k \to N$ of $1/(2k-1)$. By numerical experiment I found that the error was roughly equal to $1/(4N)$ and so used this to improve the approximation. I then computed residual errors and found them proportional to $1/N^3$. There are two series, depending on the sign of the last deleted term. In this way I developed the following two approximations, which have some semblance of asymptotic formulae:

Series approx for N even

$$\mathbf{\Sigma}_A(N) = \sum_{j=1}^{N} \frac{(-1)^{(j+1)}}{2j-1} \approx \tfrac{1}{2}\tan^{-1} N + \frac{1}{4N} - \frac{1}{9 \cdot 61\, N^3} \qquad \text{where } \frac{1}{9 \cdot 61} \approx \frac{541}{5199}.$$

As examples of the accuracy, the errors for some values of N are

$N = 10$	$1.26E-07$
$N = 20$	$-6.6E-09$
$N = 100$	$-1.1E-10$
$N = 400$	$-1.4E-11$
$N = 2000$	$-1.4E-14$
$N = 10000$	$1.1E-16$

The equivalent approximation for N odd is

$$\mathbf{\Sigma}_B(N) = \sum_{j=1}^{N} \frac{(-1)^{(j+1)}}{2j-1} \approx \tfrac{1}{2}\tan^{-1} N + \frac{3}{4N} - \frac{0.22917}{N^3} + \frac{1}{500N^4} \qquad \text{where } 0 \cdot 22917 \approx \frac{11}{48}.$$

Examples of errors are

$N = 11$	$9 \cdot 57E-07$
$N = 21$	$3.35E-08$
$N = 101$	$9.6E-13$
$N = 401$	$-8.4E-15$
$N = 2001$	$2.2E-16$
$N = 10001$	$1.1E-16$

In general

$$S_{6k+7} = \frac{\pi}{4} - \mathbf{\Sigma}_A(3k+3) \text{ for } k \text{ odd}$$
$$-S_{6k+7} = \frac{\pi}{4} - \mathbf{\Sigma}_B(3k+3) \text{ for } k \text{ even}$$

With $k = 1$

$$S_{13} \approx \frac{\pi}{4} - \tfrac{1}{2}\tan^{-1} 6 - 0 \cdot 0411849 = 0 \cdot 0413894,$$

in encouraging agreement with the true value $0 \cdot 0413866$. Using both the N-even and N-odd approximations, the error in $S_{13}/6 - S_{19}/20$ is 6×10^{-7}. For a general odd value of k the sum of the pair of terms corresponding to k and $k+1$ is approximately

$$\frac{1}{8k(k+1)(2k+1)(2k+3)}\Big[(4k^2+6k+3)\pi$$

$$-2(k+1)(2k+3)\tan^{-1}(3k+3)-2k(2k+1)\tan^{-1}(3k+6)-\frac{P(k)}{(3(k+1)^2(k+2)^4}\Big]$$

where

$$P(k)=8k^7+74k^6+286k^5+600k^4+749k^3+565k^2+242k+46.$$

This involves inverse tangents but these in turn have the asymptotic form

$$\tan^{-1}N\asymp\frac{\pi}{2}-\frac{1}{N}+\frac{1}{3\,N^3}-\frac{1}{5\,N^5}+\frac{1}{7\,N^7}+$$

With this the sum of a pair of adjacent terms can be approximated closely by a rational fraction in k alone. For accuracy I have evaluated this as a (degree 16 polynomial)/(degree 20 polynomial) which begins

$$\frac{612360\,k^{16}+10379880\,k^{15}+79510410\,k^{14}+416871756\,k^{13}+\cdots}{9797760\,k^{20}+205752960\,k^{19}+1988945280\,k^{18}+11727918720\,k^{17}+\cdots}\qquad Eq.12.7$$

Using this, the approximate value of $S_{73}/(22\times 23)-S_{79}/(24\times 25)$ is 3.27×10^{-6} compared with the true value of $3\cdot 04\times 10^{-6}$. For very large values of k this can be approximated further by a Taylor series in $1/k$. (Here, to control untidy fractions, I have approximated the higher coefficients by their simpler convergents in continued fraction expansion.) The first few terms are

$$\frac{1}{16\,k^4}-\frac{1}{4\,k^5}+\frac{23}{31\,k^6}+\frac{88}{25\,k^7}-\frac{41}{6\,k^8}+\frac{66}{5\,k^9}-\frac{101}{4\,k^{10}}+\cdots\qquad Eq12.8$$

Note that although the coefficients increase, successive terms decrease for values of k greater than about 5. This, therefore, simplifies the form of the asymptotic approximation, while the loss of accuracy from using this simplified power series over the rational approximation, Eq 12.7, is small for k greater than about 7.

In summary, for a given large value of k Eq. 12.8 gives the approximate sum of the pair of terms for k and $k+1$ in the series involving the S_{6k+7} which make up **S**. In calculation, k steps by 2.

Using this asymptotic approximation I have summed the terms in the S_{6k+7} series from $k=13$ effectively to infinity, and find the value $4\cdot 826\times 10^{-6}$. This is to be added to the exact values for the first 12 terms of the series, which was found above to be $0\cdot 07181312511\,\pi-0\cdot 21975571623=0\cdot 00585187006$. Together they give **S** as 0·0058567. Nevertheless, for accurate numerical work I will return to using the directly calculated value of $\mathbf{S}=0\cdot 0058565761$ quoted at the conclusion of §12.3. Add to this the other terms in 3**K** from Eq.12.6 and we have, numerically, $3\mathbf{K}=-0\cdot 02277137391$, as required to match the value of the log integral obtained by quadrature.

12.5 Collecting results for **J** *and* **K**

We are now in a position to determine our concluding value of the integral, using the **J** and **K** constituent sub-series. We have split

$$\int_0^1\frac{\ln(x^3+1)}{x^2+1}\,dx\text{ into }\mathbf{J}+3\mathbf{K}\ .$$

From §12.2 Eq 12.4 we found

$$\mathbf{J} = \left(\frac{\pi}{3} - \frac{13}{20}\sqrt{3}\right)\ln(2+\sqrt{3}) \ + \ \frac{\pi}{2}\ln 2 \ - \ \frac{4\pi}{15} = 0\cdot 1474753961.$$

$$3\mathbf{K} = -3\mathbf{G} + \left(\frac{\pi}{4} - \frac{13}{30}\right)\ln 2 - \frac{3\pi}{8} + \frac{274}{75} + \left[\frac{S_{13}}{2.3} - \frac{S_{19}}{4.5} + \frac{S_{25}}{6.7} - \cdots\right]$$

$$\int_0^1 \frac{\ln(x^3+1)}{x^2+1}\,dx = \left(\frac{\pi}{3} - \frac{13}{20}\sqrt{3}\right)\ln(2+\sqrt{3}) + \left(\frac{3\pi}{4} - \frac{13}{30}\right)\ln 2 - 3\mathbf{G} - \frac{77}{120}\pi + \frac{274}{75} + \mathbf{S}. \qquad Eq.12.7$$

$$= 0\cdot 1247040222$$

in excellent agreement with numerical quadrature.

In closing this section we might pause to consider what is the meaning of the above integral, which seems not to be expressible in closed form. I have obtained a solution of sorts, but it is probably not unique because there will be other ways of splitting up the infinite series involved and summing the resulting sub-series. But the fact that I have had to resort to a series solution in the first place is strong evidence that this integral cannot be evaluated by contour integration. Perhaps the only real integrals which can be evaluated analytically by contour integration are those which have a closed form, and not ones which require effectively a new functional series to be defined. It therefore raises for me the larger question of the relationship between contour integrals and series evaluation of real integrals. I invite comments from readers.

Finally, a comment about Putnam-type integrals involving fractional powers. I showed in §8.1 that the substitution $u^2 = x$ makes

$$\int_0^\infty \frac{\ln(x^{k/2}+1)}{x^2+1}\,dx = 2\int_0^\infty \frac{u\ln(u^k+1)}{u^4+1}\,du$$

and we are interested in cases where k an odd integer. I doubt whether any of these cases can be evaluated in closed form, and hence whether they can be performed by contour integration, even for $k = 1$. For $k \geq 3$ k is coprime to the 4 in the denominator so it is not possible to have two sections of contour on which both $u^k + 1$ and $u^4 + 1$ are simultaneously real, which seems to be a condition for evaluating these integrals. I leave this open as a challenge to any reader bold enough!

This concludes my hobby investigation of integrals of the logarithm, generalising the Putnam challenge integral.

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Appendix: A proof of the Putnam integral $\int_0^1 \frac{\ln(x+1)}{x^2+1}\,dx$ using infinite series

There is the well known MacLaurin-Taylor series for $\ln(x+1)$ valid for $-1 < x \leq 1$:

$$\ln(x+1) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots \qquad Eq.A1$$

It seems reasonable to divide by x^2+1 and integrate term by term, then try to identify the resulting series with the product of a series of π and a series for $\ln 2$. I tried this approach first; here is how it goes.

The series for $\ln 2$ comes by putting $x = 1$ in Eq. A1:

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots \qquad Eq.A2$$

The series for π comes from the Gregory series for $\arctan x$, using $\arctan 1 = \frac{\pi}{4}$:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$$

Use the binomial theorem to write:

$$(x^2+1)^{-1} = 1 - x^2 + x^4 - x^6 + \cdots, \qquad |x| < 1$$

so the integrand is the product of two infinite series:

$$\frac{\ln(x+1)}{x^2+1} = \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots\right)\left(1 - x^2 + x^4 - x^6 + \cdots\right)$$

In multiplying this out it is useful to work in columns of powers of x and record only the coefficients of x:

x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9	x^{10}
1		-1		1		-1		1	
	$-\frac{1}{2}$		$\frac{1}{2}$		$-\frac{1}{2}$		$\frac{1}{2}$		$-\frac{1}{2}$
		$\frac{1}{3}$		$-\frac{1}{3}$		$\frac{1}{3}$		$-\frac{1}{3}$	
			$-\frac{1}{4}$		$\frac{1}{4}$		$-\frac{1}{4}$		$\frac{1}{4}$
				$\frac{1}{5}$		$-\frac{1}{5}$		$\frac{1}{4}$	
					$-\frac{1}{6}$		$\frac{1}{6}$		$-\frac{1}{6}$

Integrate term by term and again record the coefficients of x^n in a triangular table:

x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9	x^{10}	x^{11}	
	$\frac{1}{2}$		$-\frac{1}{4}$		$\frac{1}{6}$		$-\frac{1}{8}$		$\frac{1}{10}$		$\cdots$
		$-\frac{1}{2.3}$		$\frac{1}{2.5}$		$-\frac{1}{2.7}$		$\frac{1}{2.9}$		$-\frac{1}{2.11}$	$\cdots$
			$\frac{1}{3.4}$		$-\frac{1}{3.6}$		$\frac{1}{3.8}$		$-\frac{1}{3.10}$		$\cdots$
				$-\frac{1}{4.5}$		$\frac{1}{4.7}$		$-\frac{1}{4.9}$		$\frac{1}{4.11}$	$\cdots$
					$\frac{1}{5.6}$		$-\frac{1}{5.8}$		$\frac{1}{5.10}$		$\cdots$
						$-\frac{1}{6.7}$		$\frac{1}{6.9}$		$-\frac{1}{6.11}$	$\cdots$
							$\cdots$	$\cdots$	$\cdots$		

When the limits of integration, 0 to 1, are substituted, this table of coefficients becomes a table of constants which are to be added to give the value of the integral.

From this table produce another table as follows. Starting at each non-zero value in the top row in turn, read down the column then right across the next lowest row in this fashion $\downarrow\rightarrow$. There is a common factor in each such path; for instance the path $\frac{1}{6}, -\frac{1}{3.6}, \frac{1}{5.6}, -\frac{1}{6.7}, \frac{1}{6.9}, -\frac{1}{6.11}, \ldots$ has the factor $\frac{1}{6} = \frac{1}{2} \times \frac{1}{3}$. Take out the factor of $\frac{1}{2}$ which is in every term. Hence arrive at the table

$$\begin{array}{llllllllll}
1 & [& 1 & -\frac{1}{3} & +\frac{1}{5} & -\frac{1}{7} & +\frac{1}{9} & -\frac{1}{11} & \cdots &] \\
-\frac{1}{2} & [& 1 & -\frac{1}{3} & +\frac{1}{5} & -\frac{1}{7} & +\frac{1}{9} & -\frac{1}{11} & \cdots &] \\
+\frac{1}{3} & [& 1 & -\frac{1}{3} & +\frac{1}{5} & -\frac{1}{7} & +\frac{1}{9} & -\frac{1}{11} & \cdots &] \\
-\frac{1}{4} & [& 1 & -\frac{1}{3} & +\frac{1}{5} & -\frac{1}{7} & +\frac{1}{9} & -\frac{1}{11} & \cdots &] \\
+\frac{1}{5} & [& 1 & -\frac{1}{3} & +\frac{1}{5} & -\frac{1}{7} & +\frac{1}{9} & -\frac{1}{11} & \cdots &] \\
\cdots & & & & & & & & &
\end{array}$$

The bracket, repeated every row, is $\frac{\pi}{4}$ and the vertical column of multiplying factors is $\ln 2$. With the factor of $\frac{1}{2}$ taken above, the definite integral is evaluated as $\frac{\pi}{8} \ln 2$. Q.E.D. ¶

Strictly, we now have to prove that this manipulation of infinite series is valid. But that's another story ....

¶ *Quod erat demonstrandum* = 'which was to be proved '; all old maths books used this Latin expression to signal the successful completion of a proof.