Q: Evaluate the indefinite integral $\int\sqrt{\tan x}\,dx$ and the definite integrals over the intervals [0, π/4] and [0,π/2].

1 Overview of function

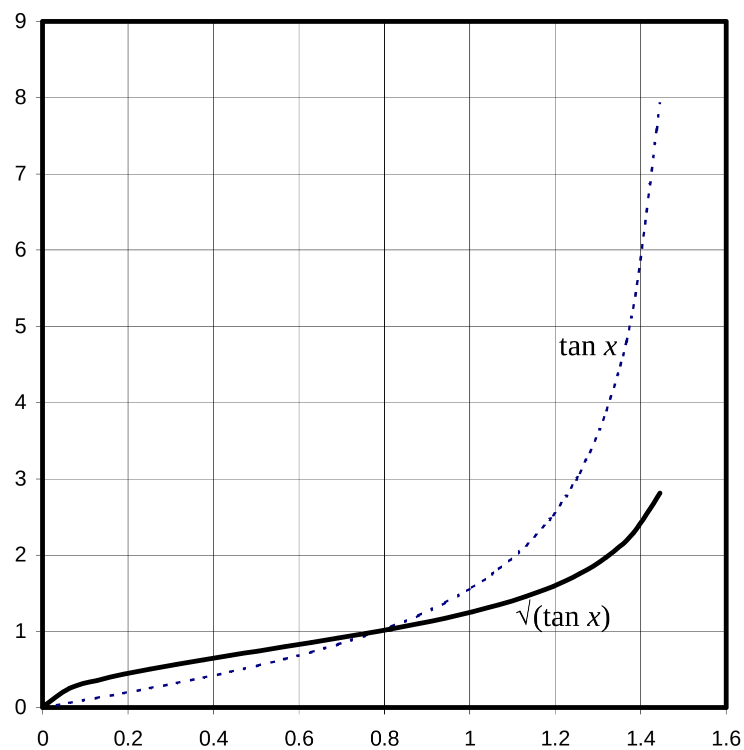
It is helpful first to have a picture of the integrand. Figure 1 shows $\sqrt{\tan x}$ compared with tan x from $x = 0$ to near π/2. For $x < 0$ and $x > \pi/2$, tan x is negative and the square root complex. There is a singularity at $x = \pi/2$. Writing $x = \pi/2 - y$, we have

$$\tan x = \cos y / \sin y = 1/\tan y,$$

which is close to $1/y$ for y small. Thus the singularity in $\sqrt{\tan x}$ diverges as $1/\sqrt{\pi/2 - x}$. We also see from this symmetry that

$$\int_0^{\pi/4}\sqrt{\tan x}\,dx = \int_{\pi/4}^{\pi/2}\frac{dx}{\sqrt{\tan x}} \quad \text{and} \quad \int_0^{\pi/2}\sqrt{\tan x}\,dx = \int_0^{\pi/2}\frac{dx}{\sqrt{\tan x}}$$

Figure 1: The integrand $\sqrt{\tan x}$ compared with tan x over most of the range of interest



2. Indefinite integral

To perform the indefinite integration we look for a substitution which will get rid of the square root and convert the trigonometric function into a polynomial or rational function, in which we hope to recognise something more familiar. $\sqrt{\tan x} = u$ is an obvious option to try. Then $du/dx = (1+\tan^2 x)/(2\sqrt{\tan x}) = (1+u^4)/(2u)$ and

$$\int\sqrt{\tan x}.dx = \int\frac{2u^2}{1+u^4}du\,. \qquad \text{Eq 1)}$$

To reduce this consider partial fractions. The denominator does not factorise in reals, but is $(1 + iu^2)(1 - iu^2)$ if we allow complex numbers. The integrand separates into

$$\frac{2u^2}{1+u^4} = \frac{i}{1+iu^2} - \frac{i}{1-iu^2}\ . \qquad \text{Eq 2)}$$

Apart from the *i* in the denominators, we now have two standard integrals which yield inverse trigonometric and hyperbolic functions or their logarithm equivalents:

$$\int \frac{dv}{1+v^2} = \tan^{-1} v = \frac{-i}{2}\ln\left(\frac{1+iv}{1-iv}\right) \quad \text{and}$$

$$\int \frac{dv}{1-v^2} = \tanh^{-1} v = \frac{1}{2}\ln\left(\frac{1+v}{1-v}\right) \qquad \text{Eq 3)}$$

The high symmetry between these pairs of equations arises from the tan and tanh functions linked over the complex plane by the relations

$\tan ix = i \tanh$, $\tanh ix = i \tan x$, $\tan^{-1} ix = i \tanh^{-1} x$, $\tanh^{-1} ix = i \tan^{-1} x$.

The above equations can be converted into the more general forms

$$\int \frac{a}{1+au^2} du = \sqrt{a}\tan^{-1}(u\sqrt{a}) = \frac{-i\sqrt{a}}{2}\ln\left(\frac{1+iu\sqrt{a}}{1-iu\sqrt{a}}\right) \quad \text{and}$$

$$\int \frac{a}{1-au^2} du = \sqrt{a}\tanh^{-1}(u\sqrt{a}) = \frac{\sqrt{a}}{2}\ln\left(\frac{1+u\sqrt{a}}{1-u\sqrt{a}}\right) \qquad \text{Eq 4)}$$

Now let $a = i$, $\sqrt{a} = (1+i)/\sqrt{2}$ and use $\ln(p+iq) = \ln\sqrt{(p^2+q^2)} + i\tan^{-1}(q/p)$

$$\int \frac{i.du}{1+iu^2} = \frac{-1+i}{2\sqrt{2}}.\left(\ln(\sqrt{2}+u-iu) - \ln(\sqrt{2}-u+iu)\right) \qquad \text{Eq 5)}$$

$$= \frac{-1+i}{4\sqrt{2}}.\left(\ln(u^2+1+u\sqrt{2}) - 2i\tan^{-1}\left(\frac{u}{\sqrt{2}+u}\right) - \ln(u^2+1-u\sqrt{2}) - 2i\tan^{-1}\left(\frac{u}{\sqrt{2}-u}\right)\right)$$

and

$$\int \frac{i.du}{1-iu^2} = \frac{1+i}{2\sqrt{2}}.\left(\ln(\sqrt{2}+u+iu) - \ln(\sqrt{2}-u-iu)\right) \qquad \text{Eq 6)}$$

$$= \frac{1+i}{4\sqrt{2}}.\left(\ln(u^2+1+u\sqrt{2}) + 2i\tan^{-1}\left(\frac{u}{\sqrt{2}+u}\right) - \ln(u^2+1-u\sqrt{2}) + 2i\tan^{-1}\left(\frac{u}{\sqrt{2}-u}\right)\right)$$

On multiplying out the complex products Eq 5) and 6) and taking their difference in accordance with Eq 2), all complex terms cancel and we obtain

$$\int \sqrt{\tan x}.dx = \int \frac{2u^2}{1+u^4}du =$$

$$\frac{1}{2\sqrt{2}}\left(\ln\left(\frac{1+u^2-u\sqrt{2}}{1+u^2+u\sqrt{2}}\right)+2\tan^{-1}\left(\frac{u}{\sqrt{2}+u}\right)+2\tan^{-1}\left(\frac{u}{\sqrt{2}-u}\right)\right) \qquad \text{Eq 7)}$$

where $u = \tan x$. This is the required indefinite integral. The second arctangent term needs some care in interpretation because of the singularity at $u = \sqrt{2}$. The arctangent is a multi-valued with branches at separation π. At $u = \sqrt{2}$ the function jumps from one branch to another. For $u > \sqrt{2}$ the interpretation is

$$\tan^{-1}\left(\frac{u}{\sqrt{2}-u}\right) = \pi - \tan^{-1}\left(\left|\frac{u}{\sqrt{2}-u}\right|\right) \quad . \qquad \text{Eq 8)}$$

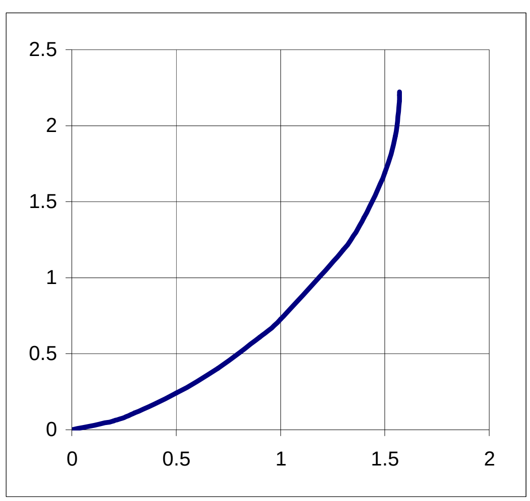
2 *Definite integrals and numerical values*

We can gain some confidence in the correctness of Eq 7) by comparing it with a selection of values obtained by numerical integration. For $x = [0, \pi/4]$, $u = [0, 1]$ and the numeric estimate is 0.487491, which compares with 0.4874955 from direct evaluation of Eq 7. For $x = [0, 1]$, $u = [0, 1.24796]$ and the numeric estimate is 0.727291, compared with 0.727298 from Eq 7). For u = [0, 1.5] the numeric estimate is 0.93568 and the value from Eqs.. 7) and 8) is 0.93569.

For $x = [0, \pi/2]$ somewhat special numerical techniques are required to cope with the singularity (essentially Gaussian-type quadrature using orthogonal polynomials with the $1/\sqrt{x}$ singularity 'built in' to the quadrature). With such appropriate measures the estimate 2.22148 has been obtained. This is very close to $\pi/\sqrt{2}$. To obtain this value analytically we take the limit of $u \to \infty$. The logarithm in Eq 7) tends to 0. The arctangent terms tend to $2\tan^{-1}(1)$ and $2\pi - 2\tan^{-1}(1)$ respectively. The result is $\pi/\sqrt{2}$ for the integral $[0, \pi/2]$, in agreement with the numerical estimate.

Figure 2 below is a graph of the integral.

$$\int_0^x \sqrt{\tan t}.dt$$



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