Q: a) From the first 200 natural numbers 101 of them are arbitrarily chosen. Prove that among the numbers chosen there exists a pair such that one divides the other.

b) Prove that if 100 numbers are chosen from the first 200 natural numbers and include a number less than 16, then one of them is divisible by another.

c) Generalise this to choosing n+1 numbers from 1, 2, ... 2n.

We approach this problem in two ways. First, by an existence theorem using the pigeon hole principle, and second by an analysis of sets of mutually indivisible integers.

We will call pairs of integers, within the range 1 to N, for which one will not divide the other 'mutually indivisible'. The concept is meaningful only for N finite.

1. Existence: Pigeon Hole Principle

1.1 Question a)

In the range 1 to 200 there are 100 odd numbers, so if all 100 were drawn and then a 101^{st} odd number drawn, it must be equal to one already selected. This pair will divide one another with quotient 1.

A singly even number (one divisible by 2 but not by 4) will equal 2k for odd $k \in \{1, 3, ... 99\}$. We can regard the draw being done by carrying out a primary selection from an index set of 100 odd numbers $\{k\} = \{1, 3, 5 ....199\}$ and then, for arbitrary $k \leq 99$, choosing 2k instead of k. Again if 101 values of k are selected, at least two must be the same and now the numbers indexed to them, whether odd or even, must divide one another with quotient 1 or 2 respectively.

The principle of indexing to k can be extended to even numbers with factors 2^r where $r \geq 2$. Thus $12 = 2^2.3$ so k(12) is 3, and $24 = 2^3.3$ is also indexed by k of 3. All integers indexed by a given k are of the form $2^r.k$ and so any pair of them has a quotient which is a power of 2. Since any 101 values of k must have two the same, the corresponding integers divide one another.

Table 1 below illustrates the construction for the case of 11 integers selected from 20.

Table 1 : indexing of even numbers by odd values , k for N = 20

indexing set k	1	3	5	7	9	11	13	15	17	19
	2	6	10	14	18					
even	4	12	20							
alternatives to k	8									
	16									

1.2 Question b)

Table 2 below shows the left hand end of the corresponding table for N = 200. We are now given that there is seeded a number less than 16, corresponding to an odd index value in the range 1 to 15. Next select 99 other index values. If two of these k are the same, divisibility of a pair of integers is assured by the argument above.

Table 2 : indexing of even numbers by odd values , k, for N = 200

indexing set k	**1**	**3**	**5**	**7**	**9**	**11**	**13**	**15**	17	19	21	23	25	27	29
	2	**6**	**10**	14	18	22	26	30	34	38	42	46	50	54	58
	4	**12**	20	28	36	44	52	60	68	76	84	92	100	108	116
even	**8**	24	40	56	72	88	104	120	136	152	168	184	200		
alternatives to k	16	48	80	112	144	176									
	32	96	160												
	64	192													
	128														

If all k are different (as is now possible), we find that at least one of the small numbers will divide a larger. This is because one number from every column is present and also several pairs of numbers in the same row are mutually divisible. Table 3 lists some divisible combinations when there is only one integer ≤ 15. (This is the more challenging situation.) For each integer one of the divided numbers must be present.

Table 3 : guaranteed divisible pairs for integer < 16

integer	**k**	**divided number**
1	1	any other integer
2	1	any even, or else 61 183, or 63 189, say
3	3	any multiple of 3 *e.g.* {18, 36, 72}, or large multiple *e.g.* 117
4	1	any multiple of 4 *e.g.* {20, 40, 80, 160}
5	5	{30, 60,120} or multiple of 5 *e.g.* 165
6	3	{18, 36, 72, 144} and {30, 60, 120}
7	7	any multiple of 21 or 63
8	1	{24, 48, 96} or {40, 80, 160}
9	9	any multiple of 27 or 81
10	5	{30, 60,120}
11	11	any multiple of 33 or 55
12	3	{36, 72, 144} or 18 {54, 108} or 27 135
13	13	any multiple of 39 or 65
14	7	{42, 84, 168} or 21 {63, 126}
15	15	any multiple of 45 or 75

Some explanation of the column 'divided number' in the most involved case of integer 12 is as follows. By our assumption, only 12 is less than 16. Under index 9 in Table 2 the possible integers are 18, 36, 72 or 144. 12 divides all but 18, so if 18 is the one present, it will divide 54 or 108 under index 27. If 27 itself is present rather than either of 54 or 108, it will divide 135, which must be present since all k are represented.

The above discussion answers parts a) and b) of the question. The pigeon hole argument generalises immediately to N integers, by replacing '200' in the above by N.

2 Analysis of mutually indivisible numbers

The pigeon hole principle, though powerful in setting bounds, gives little insight into which integers will be in the largest sets of mutually indivisible integers. Moreover, we have not yet determined whether it gives an accurate value (100 in the case of N = 200) for the size of the largest set of mutually indivisible integers, nor how this value might vary with the smallest integer in the set, q. Hence this second approach to the problem examines sets of mutually indivisible integers. The goal would be to specify those particular integers which occur in the largest set of mutually indivisible integers as a function of the smallest integer, q, in the selection; and hence to determine the cardinality of this set, m(q, N). The strategy is to contrive sets of numbers which avoid divisibility as far as possible.

2.1 Basic considerations

The following considerations apply:

1) q = 1 is not acceptable because 1 divides every other integer. (Note that the pigeon hole principle counts trivial divisions by 1 and by an integer with itself.)

2) For integer division the ratio of the integers must be greater than 2. Hence all integers j in the set S_0 $\{j: q \le j \le 2q - 1\}$ are mutually indivisible. As a special case, if q = N/2 + 1 to N (N even) or (N+1)/2 to N (N odd) there will be N/2 (N even) or (N+1)/2 (N odd) consecutive mutually indivisible integers. Hence the pigeon hole principle does give the correct value of m(q, N) for q = N/2 + 1. Indeed, applying this to the given case of N = 200, note immediately that the 100 integers 101 to 200 inclusive are mutually indivisible.

3) By the fundamental theorem of arithmetic every integer is either a prime or a unique product of primes. No product of two primes can divide another product of two primes, though of course it can be divided by either of the constituent primes. In general, a product of r primes from some set G cannot divide a product of r – s, s > 0, primes drawn from the same set. We call a product of r primes a product of degree r.

4) Hence, products of primes of different degree will be mutually indivisible, and a set of degree r must (subject to certain latitude outlined below) be kept disjoint from a set of another degree. The 'building blocks' of sets of mutually indivisible integers in the range q to N are therefore the sets S_1, S_2, S_3., …of degree 1, 2, 3, .. etc.

2.2 The basic mutually indivisible sets

Before discussing the complexity introduced by taking the partial union of sets of different degree, we illustrate the discussion by tabulating S_1, S_2, S_3 for the case of N = 200.

S_0: Recall that this is the set from q to 2q –1. These numbers are either prime or products of primes less than q.

S_1: These are 46 primes between 2 and 199. They are

Table 4 : primes less than 200

2 3 5 7 11 13 17 19 23 29 31 37 41 43 47 53 59 61
67 71 73 79 83 89 97 101 103 107 109 113 127 131 137
139 149 151 157 163 167 173 179 181 191 193 197 199

S_2: There are 62 products of 2 primes in the range. We define the generating set G_2 of these as the primes up to 97, which is the largest prime p so that 2p < 200.

Table 5: products of 2 primes

	2	3	5	7	11	13	17	19	23	29	31	37	41	43	47	53	59	61	67	71	73	79	83	89	97
2	4	6	10	14	22	26	34	38	46	58	62	74	82	86	94	106	118	122	134	142	146	158	166	178	194
3		9	15	21	33	39	51	57	69	87	93	111	123	129	141	159	177	183							
5			25	35	55	65	85	95	115	145	155	185													
7				49	77	91	119	133	161																
11					121	143	187																		
13						169																			

S_3: There are 48 products of three primes. The generating set G_3 of these is all primes up to 47, which is the largest prime p such that $2^2p < 200$.

Table 6 : products of 3 primes

	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	Factor
	2	3	5	7	11	13	17	19	23	29	31	37	41	43	47	Factor
2	8	12	20	28	44	52	68	76	92	116	124	148	164	172	188	Product

	3	3	3	3	3	3	3	3	3	3
	3	5	7	11	13	17	19	23	29	31
2	18	30	42	66	78	102	114	138	174	186
3	27	45	63	99	117	153	171			

	7	7	7
	7	11	13
2	98	154	182
3	147		

	5	5	5	5	5	5	5
	5	7	11	13	17	19	23
2	50	70	110	130	170	190	
3	75	105	165	195			
5	125	175					

S_4: There are only 25 products of four primes in the range.

Table 7 : products of 4 primes

		2	2	2	2	2	2	2	2	2	3	3	3
		2	3	5	7	11	13	17	19	23	3	5	7
2	2	16	24	40	56	88	104	136	152	184			
2	3		36	60	84	132	156						
2	5			100	140								
2	7				196								
3	3		54	90	126	198					81	135	189
3	5			150									

2.3 Constructing a large set of mutually indivisible integers

We are seeking to identify the maximal set of indivisibles for a given q, and will attempt to approximate to this by the following strategy. Start with a set of r (high) degree composite numbers derived from the smallest primes, and then adjoin integers of degree r –1 made from the rather larger primes, then adjoin integers of degree r – 2 composed of even larger primes, and so on until only the largest single primes (degree 1) are adjoined. Finally check that all eligible integers in the set S_0 are adjoined.

The rationale is that, for the given values of q and N (here 200), there will be a degree r of composite integers made from only the smallest primes which produces a larger number of integers in the range q to 200 than is produced through products of another degree. However these smaller primes are effectively ‘used up’ in generating these degree r integers, in the sense that composite numbers of lower degree made from them would divide those of degree r. *A fortiori*, it is not maximal to have small single primes in the final set.

The above considerations have not been developed to the status of an algorithm, but the illustration below for the case q = 8 hopefully gives some insight into what a maximal set could look like, and allows an alternative estimate of m(q, N) to compare with the value of 99 obtained using the pigeon hole principle, in answer to part b) of the question.

2.4 Illustration: q =8

With q = 8, the sets are

S_0 : 8, 9, 10, 11, 12, 13, 14, 15
S_1: all primes ≥ 11
S_2 : all values in Table 5 except 4 and 6
S_3 : all values in Table 6
S_4 : all values in Table 7.

In S_4 we see there are 5 degree-4 integers composed from 2 and 3 : {16, 24, 36, 54, 81} compared with 4 degree-3 integers in S_3, namely {8, 12, 18, 27}. However q = 8 so we are obliged to accept the four in S_3.

The next step is to make the optimum selections from S_3 and S_2. Table 8 below lists the number of degree-3 and degree-2 integers for combination of primes from {2, 3} to {2, ... 47}. This suggests that it is optimum to use the degree-3 integers generated with primes up to 47. This is, in fact, all the 48 integers in S4 (Table 6).

Table 8: comparison of number of degree 4 and degree-3 integers generated by small primes

upper prime	**No. of composites in S_3**	**No. of composites in S_2**
11	25	13
13	30	19
17	34	24
19	38	28
23	40	32
29	42	35
31	44	38
37	45	41
41	46	43
43	47	45
47	48	47
53	48	49

We now add those degree-2 integers from S_2 which are mutually indivisible with those in S_3. These are

i) the composites made with primes > 47 and
ii) those composites of primes ≤ 47 that correspond to degree-4 integers too large for the range *i.e.* >200. For example, 2.5.23 = 230 so cannot be included, but this leaves 5.23 = 115 available for inclusion.

Regarding point i) there is little merit in including the even numbers 2.67, 2.71, etc. since they are equal in number to the primes 67, 71 ... in S_1.

Regarding point 2, the following integers 15 can be adjoined (here laid out as in Table 5):

							111	123	129	141
				115	145	155	185			
		119	133	161						
121	143	187								
	169									

Taking stock, the putative ‘maximal set’ so far has 48 + 6 + 15 = 69 integers.

The next step is to adjoin the 28 primes > 61. This gives 97 integers.

The final step is to consider whether any integers in S_0 can be adjoined. Clearly none can.

2.5 Summary

Our attempt at creating an explicit maximal set for q = 8 has produced the following 97 integers:

8 12 18 20 27 28 30 42 44 45 50 52 63 66 67 68 70 71 73 75 76 78 79 83 89 92
97 98 99 101 102 103 105 106 107 109 110 111 113 114 115 116 117 118 119 121 122
123 124 125 127 129 130 131 133 137 138 139 141 143 145 147 148 149 151 153 154
155 157 159 161 163 164 165 167 169 170 171 172 173 174 175 177 179 181 182 183
185 186 187 188 190 191 193 195 197 199

As a separate exercise, you could check these by computer for mutual indivisibility, and confirm that the addition of any one extra integer will cause a pair of integers to divide. The analysis could be taken further, but it is of interest that the size of the above set is 3 less than the 100 predicted from the ‘mere’ pigeon hole principle, and 2 less than the modification for q <16 demonstrated in Section 1.2 above.

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