

Q5: Which positive integers can be expressed as the sum of three or more consecutive positive integers?

Suppose integer q is the sum of the $k+1$ integers $n, (n+1), (n+2), \dots, (n+k)$. Then

$$q = \sum_0^k n + \sum_0^k k = n(k+1) + \frac{1}{2} k(k+1) = (k+1)(n + \frac{1}{2} k)$$

If k is even, $k = 2m$, say, then $q = (2m+1)(n+m)$.

If k is odd, $k = 2m+1$ say, $q = (2m+2)(n + m + \frac{1}{2}) = (m+1)(2n + 2m + 1)$.

In both cases q has at least one odd factor. Hence q cannot be a power of 2.

If q is prime, one of the factors must be 1 and the other must be q . The case $m = 0$ for k even corresponds to the trivial case of only one term, $n = q$. $m = 0$ for k odd limits the options to q being the sum of only two consecutive numbers, n and $n+1$: e.g. $17 = 8 + 9$. We conclude that no prime can be expressed as the sum of 3 or more consecutive positive integers.

However, if we change the rules to allow some negative integers, primes can be expressed as sums of three or more consecutives. As an example,

$$17 = -7 - 6 - 5 \dots - 1 + 0 + 1 + \dots + 9.$$

Here is an algorithm for determining n and k for any chosen sum q :

- 1) Select q . No solution in positive integers is possible if q is a power of 2. If q is prime, the only non-trivial solution is $q = (q-1)/2 + (q+1)/2$.
- 2) Select an odd divisor of q . We now have two choices
- 3) Choice 1 : k is even
 - a) The odd divisor is $k+1$. Determine $k/2$.
 - b) Divide q by $k+1$. Call this j .
 - c) Determine $n = j - k/2$. If $n < 0$, select another odd divisor and loop back to stage a).
 - d) The required sequence is $n + (n+1) + \dots + (n+k)$.
- 4) Choice 2 : k is odd.
 - a) The odd divisor is $2(n+m)+1$. Divide q by this odd divisor. Call the quotient h .
 - b) $h = m+1$. Determine m and hence $k = 2m+1$.
 - c) Find n from the divisor $2(n+m)+1$ and m .
 - d) The required sequence is again $n + (n+1) + \dots + (n+k)$

Example. Find sequences of consecutive integers summing to $q = 4922$.

Note that $4922 = 2 \times 23 \times 107$.

Case i) k even. Select the odd divisor to be 23, so $k/2 = 11$.

$$j = 214.$$

$$n = 214 - 11 = 203.$$

$$\text{Thus } 4922 = 203 + 204 + 205 + \dots + (203+22) \quad (23 \text{ terms})$$

Case ii) k is odd. Select the odd divisor to be 23, so $n + m = 11$.

$$h = 214$$

$$m = 213, k = 427$$

$$n = -202$$

$$\text{Thus } 4922 = -202 - 201 - 200 - \dots + 225 \quad (428 \text{ terms})$$

Case iii) k even. This time select the odd divisor to be 107, so $k/2 = 53$.

$$j = 46.$$

$$n = 46 - 53 = -7. \text{ Either reject this option or accept non-negative integers.}$$

$$\text{Thus } 4922 = -7 - 6 - 5 \dots + 99 \quad (107 \text{ terms})$$

Essentially in these last two cases there are simply too many terms for their sum to be 4922 without some being negative.

Case iv) k odd. Select the odd divisor to be 107, so $n + m = 53$.

$$h = 46$$

$$m = 45, k = 91$$

$$n = 53 - 45 = 8$$

$$\text{Thus } 4922 = 8 + 9 + 10 + \dots + 99 \quad (92 \text{ terms}).$$

So we have achieved two valid sequences of consecutive positive integers adding up to 4922, plus two sequences of consecutives involving negative integers.

(I did not find this question very interesting, but have soldiered on none the less.)

John Coffey