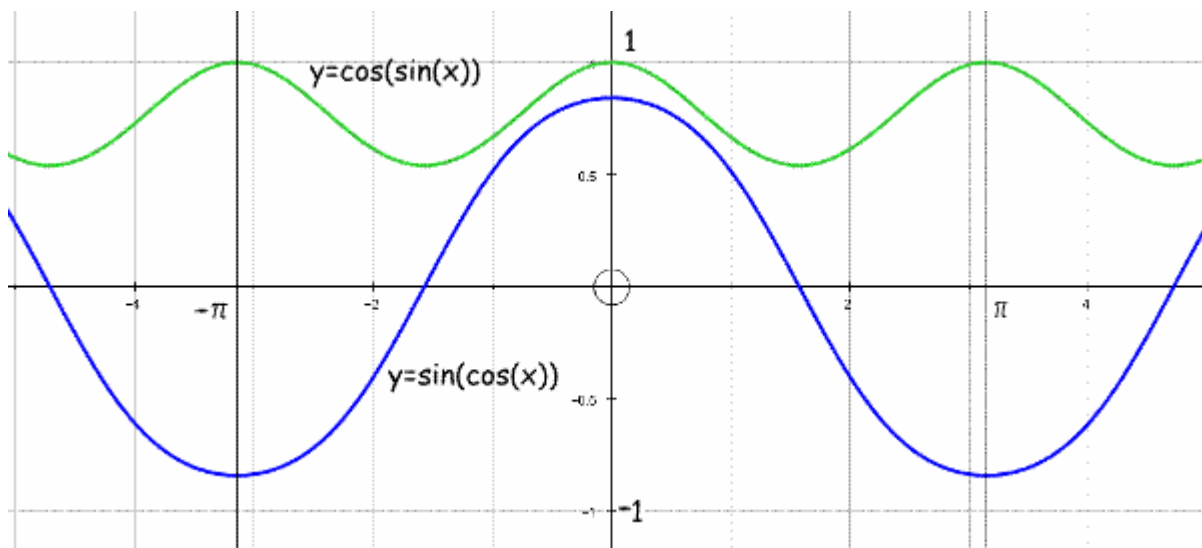


**Q6: Which is the greater,  $\cos(\sin x)$  or  $\sin(\cos x)$ ?**

‘Greater than’ implies that we are dealing with real rather than complex values of  $x$ .

Your first guess might be that there is symmetry between the two functions so one is probably just a displaced version of the other. But this is not so. We can just put the two functions into a graph plotting package and get the answer without any analysis. So here is the answer:



Now let's see if we can answer the question purely by analysis of the functions  $\sin(x)$  and  $\cos(x)$ .

Recall that  $\sin x$  and  $\cos x$  are periodic functions, period  $2\pi$ , with values lying between  $-1$  and  $+1$ . Also  $\cos x$  is an even function of  $x$ ,  $\sin x$  an odd one. Thus  $\cos(\sin x)$  lies between  $\cos(0) = 1$  and  $\cos(1) = \cos(-1) = 0.5403$ , the latter being the limit of the series:

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \dots \text{ evaluated at } x = 1.$$

Similarly  $\sin(\cos x)$  lies between  $\sin(-1) = -0.8415$  and  $\sin(1) = +0.8415$ , the latter being the limit of

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \dots \text{ evaluated at } 1.$$

Since

$$\sin(-1) < \cos(1) < \sin(1) < \cos(0), \tag{1}$$

the ranges of the two functions overlap and there is, *prima facia*, the possibility of their graphs intersecting and so of them alternating in relative greatness as  $x$  varies. We now examine this possibility.

If the graphs intersect, there exist  $y_1, y_2$  such that  $\cos y_1 = \sin(y_1 + \pi/2) = \sin y_2$  where  $y_1 = \sin(x)$ ,  $y_2 = \cos(x)$  for some  $x$ . So

$$y_2 - y_1 = \pi/2 + 2\pi k, \quad k \text{ an integer,} \tag{2}$$

Now the maximum difference between  $\sin(x)$  and  $\cos(x)$  is attained at  $x = 3\pi/4 + 2\pi n$ , when  $\sin(x)$  is  $1/\sqrt{2}$  and  $\cos(x)$  is  $-1/\sqrt{2}$ . (There is an equivalent maximum difference at  $7\pi/4$ .) The difference,  $\sqrt{2} = 1.4141\dots$ , is less than  $\pi/2 = 1.5708\dots$ , so  $y_2$  and  $y_1$  cannot be chosen to satisfy equation (2). We

conclude that the curves do **not** intersect, and so retain their relative positions for all  $x$ . Therefore, since at  $x = 0$   $\cos(\sin(0)) = \cos(0) = 1$ , and  $\sin(\cos(0)) = \sin(1) = 0.8415 < 1$ , we may conclude that  $\cos(\sin(x))$  is greater than  $\sin(\cos(x))$  for all real values of  $x$ .

We can take this a step further and determine the periods of the two functions. The graphs above show that  $\cos(\sin(x))$  has period  $\pi$  and  $\sin(\cos(x))$  has period  $2\pi$ .  $\cos(\sin(x))$  has period  $\pi$  because

i)  $\cos(-y) = \cos(y)$  for any  $y$ , and

ii)  $|\sin(x)|$  has period  $\pi$ .

So  $\cos(\sin(x + \pi)) = \cos(-\sin(x)) = \cos(\sin(x))$ .

For the other function

$$\sin(\cos(x + \pi)) = \sin(-\cos(x)) = -\sin(\cos(x)) \text{ but}$$

$$\sin(\cos(x + 2\pi)) = \sin(\cos(x)).$$

John Coffey