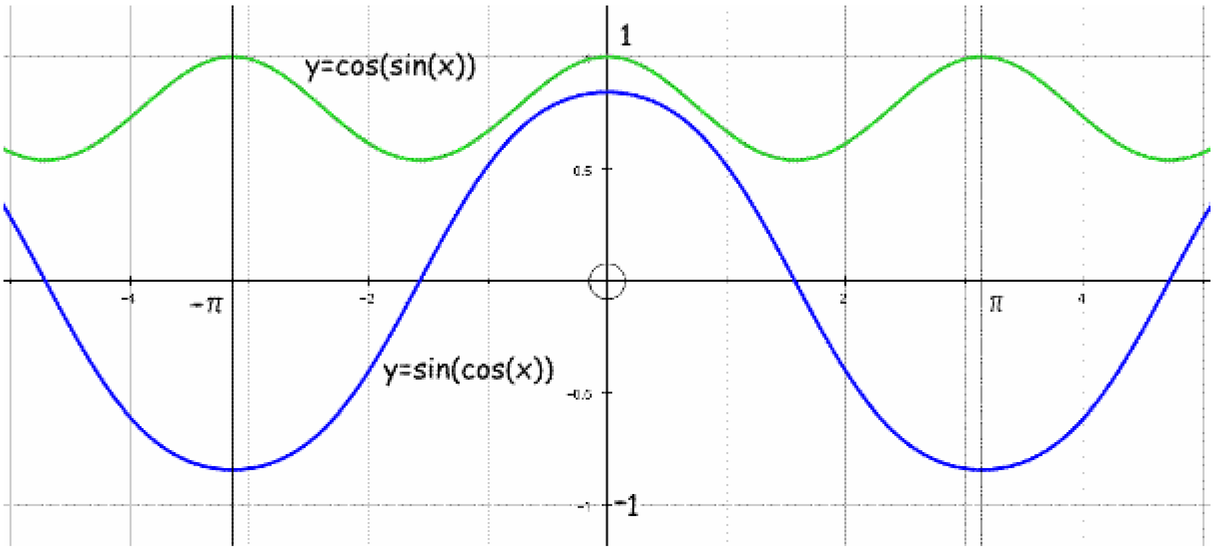
Q6: Which is the greater, cos(sin x) or sin(cos x))?

'Greater than' implies that we are dealing with real rather than complex values of x.

Your first guess might be that there is symmetry between the two functions so one is probably just a displaced version of the other. But this is not so. We can just put the two functions into a graph plotting package and get the answer without any analysis. So here is the answer:



Now let's see if we can answer the question purely by analysis of the functions sin(x) and cos(x).

Recall that sin x and cos x are periodic functions, period 2π, with values lying between –1 and +1. Also cos x is an even function of x, sin x an odd one. Thus cos(sin x) lies between cos(0) = 1 and cos(1) = cos(–1) = 0·5403, the latter being the limit of the series:

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \ldots \text{ evaluated at } x = 1.$$

Similarly sin(cos x) lies between sin(–1) = –0·8415 and sin (1) = +0·8415, the latter being the limit of

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \ldots \text{ evaluated at 1.}$$

Since

$$\sin(-1) < \cos(1) < \sin(1) < \cos(0), \qquad (1)$$

the ranges of the two functions overlap and there is, *prima facia*, the possibility of their graphs intersecting and so of them alternating in relative greatness as x varies. We now examine this possibility.

If the graphs intersect, there exist y_1, y_2 such that cos y_1 = sin (y_1 + π/2) = sin y_2 where y_1 = sin(x), y_2 = cos(x) for some x. So

$$y_2 - y_1 = \pi/2 + 2\pi k, \quad \text{k an integer}, \qquad (2)$$

Now the maximum difference between sin(x) and cos(x) is attained at $x = 3\pi/4 + 2\pi n$, when sin(x) is $1/\sqrt{2}$ and cos(x) is $-1/\sqrt{2}$. (There is an equivalent maximum difference at 7π/4.) The difference, $\sqrt{2}$ = 1.4141..., is less than π/2 = 1.5708..., so y_2 and y_1 cannot be chosen to satisfy equation (2). We conclude that the curves do ***not*** intersect, and so retain their relative positions for all x. Therefore, since at $x = 0$ $\cos(\sin(0)) = \cos(0) = 1$, and $\sin(\cos(0)) = \sin(1) = 0{\cdot}8415 < 1$, we may conclude that $\cos(\sin(x))$ is greater than $\sin(\cos(x))$ for all real values of x.

We can take this a step further and determine the periods of the two functions. The graphs above show that $\cos(\sin(x))$ has period π and $\sin(\cos(x))$ has period 2π. $\cos(\sin(x))$ has period π because

i) $\cos(-y) = \cos(y)$ for any y, and

ii) $|\sin(x)|$ has period π .

So $\cos(\sin(x + \pi)) = \cos(-\sin(x)) = \cos(\sin(x))$.

For the other function

$$\sin(\cos(x + \pi)) = \sin(-\cos(x)) = -\sin(\cos(x)) \quad \text{but}$$

$$\sin(\cos(x + 2\pi)) = \sin(\cos(x)).$$

John Coffey