Q7: Find the smallest integer n with initial digit 1 such that, if the initial digit is moved to the end, the resulting integer is 3n. Find all possible initial digits for which this can occur.

1. Solution

As a clue how to approach this problem, suppose the number N we seek is, as a decimal, 1*abc …xyz*.. Then 3N = *abc…xyz*1. To get 1 in the least significant digit of 3N, we need 3*z* = 10*K*+1 where *K* is an integer between 0 and 9, corresponding to the carry from units to tens. Now the 3 times table runs 0, 3, 6, 9, 12, 15, 18, 21, 24, 27 and the only multiple of 3 whose least digit is 1 is 7: 7×3 = 21. (This uniqueness is because 3 and 10 are coprime.) We deduce that *z* = 7 and the carry is 2. So 3*y*+2 has a least digit 7, meaning that *y* = 5. And so on until the leading digit becomes 1, when we stop.

To put this on a more formal footing, suppose that N, (expressed to base 10) has k+1 digits:

$$N = a_k.10^k + a_{k-1}.10^{k-1} + ... + 10a_1 + a_0.$$

Then $3N = 3a_k.10^k + 3a_{k-1}.10^{k-1} + ... + 30a_1 + 3a_0.$ (1)

But we are told that 3N has the property

$$3N = a_{k-1}.10^k + a_{k-2}.10^{k-1}.... + 100a_1 + 10a_0 + a_k. \qquad (2)$$

Where a simultaneous solution to (1) and (2) exists, the digits may be determined in the order $a_0, a_1, ...$ a_k by equating (1) to (2) mod 10^p, for p = 1 to k+1. Thus we have

p = 1: $3a_0 \equiv a_k \bmod 10$

p = 2: $30a_1 + 3a_0 \equiv 10a_0 + a_k \mod 100$

$30a_1 \equiv 7a_0 + a_k \mod 100$

p = 3: $300a_2 \equiv 70a_1 + 7a_0 + a_k \mod 10^3$

p = 4: $3{,}000a_3 \equiv 700\, a_2 + 70a_1 + 7a_0 + a_k \mod 10^4$

p = 5: $30{,}000a_4 \equiv 7{,}000a_3 + 700\, a_2 + 70a_1 + 7a_0 + a_k \mod 10^5$

p = 6: $300{,}000a_5 \equiv 70{,}000a_4 + 7{,}000a_3 + 700\, a_2 + 70a_1 + 7a_0 + a_k \mod 10^6$

and so on, solving for each a_j in turn until $a_j = a_k$.

In the given case of $a_k = 1$ the sequence of equations is solved as follows:

p = 1: $3a_0 \equiv 1 \bmod 10$ so $a_0 = 7$ (a unique value since 3 is coprime to 10.)

p = 2: $30a_1 \equiv 7a_0 + 1 = 50 \mod 100$

$3a_1 \equiv 5 \bmod 10$ so $a_1 = 5$.

p = 3: $300a_2 \equiv 70a_1 + 7a_0 + 1 = 400 \mod 10^3$

$3a_2 \equiv 4 \bmod 10$ so $a_2 = 8$.

p = 4: $3{,}000a_3 \equiv 6{,}000 \mod 10^4$

$3a_3 \equiv 6 \bmod 10$ so $a_3 = 2$

p = 5: $30{,}000a_4 \equiv 20{,}000 \mod 10^5$

$3a_4 \equiv 2 \bmod 10$ so $a_4 = 4$

p = 6: $300{,}000a_5 \equiv 300{,}000 \mod 10^6$ so $a_5 = 1 = a_k$

and the sequence can be terminated. Hence N = 14,2857 and 3N = 42,8571. The manner of calculation ensures that it is the lowest integer with the required property.

2. Similarity to Recurring Decimals

Here is another way to look at the numbers N and 3N. Suppose there exists a recurring number [abc …. xyz][abc …. xyz][abc …. xyz]… where the recurring section is denoted by the brackets [..]. Then N could correspond to the integer formed by [abc …. xyz] and 3N to the section [bcd …. yza], which is a cyclic permutation of [abc …. xyz]. Patterns of recurring digits occur in the decimal representation of any fraction whose denominator is coprime to 10. Some examples are

1/7 = 0·[142857]
1/11 = 0·[09]
1/13 = 0·[076923]
1/14 = 0·0[714285]
1/17 = 0·[0588235294117647]
1/19 = 0·[052631578947368421]
1/28 = 0·03[571428]

You will notice that 142857 obtained from 1/7 is the integer N found is §1 above. Multiples of 1/7 have these decimal representation :

1/7 = 0·[142857]	4/7 = 0·[571428]
2/7 = 0·[285714]	5/7 = 0·[714285]
3/7 = 0·[428571]	6/7 = 0·[857142]

Table 1

The reason that each is a cyclic permutation of the another is as follows. We are dividing 7 into 1·0000000…. so the remainder at each step of the division process produces a multiple of 10 at the next stage: *e.g.* . 10 ÷ 7 = 1 rem 3, then 30 ÷ 7 = 4 rem 2, etc. 7 is coprime to 10 so the remainder on division of 10*K*, *K* from 0 to 6, is unique. Once one remainder is specified, the next remainder is uniquely implied, so exactly the same unending chain of quotients and remainders is produced from any starting point.

The fact that there are 6 six recurring digits on division by 7 arises from the cyclic group of integer remainders (3, 2, 6, 4, 5, 1) generated by 10^p mod 7. To each remainder corresponds a quotient, so there is a cycle of six digits in the recurring decimal for 1/7.

3. Note on recursion lengths of decimal representations of fractions

1/7 achieves the maximum possible order of 7–1 = 6 to the cyclic group. As a tangential comment, examination of the length of the recursion sequence in decimal representations of 1/n, n an integer, shows that relatively few achieve their maximum possible order of (n –1); examples are the primes 7, 17, 19, 23, 29, 47, 59, 61, 97 and 193. The recursion sequence for the larger numbers such as 193 consists of a string of pseudo-random digits and therefore is a type of random number generator.

Some other integers have recursion lengths less than the maximum (n–1), but which nevertheless divide n–1. For example 1/13 has recursion length 6 which divides 12, and 1/37 has length 3 which divides 36. Some even denominators have length with divide (n/2 – 1) *e.g.* 1/34 has length 34/2–1 = 16 and 1/58 has recursion length 58/2 – 1 = 28. Others have a more complex relation to n: *e.g.* 1/49 has recursion length 42 and 1/169 has length 78.

Though 1/13 has a recursion length of 6 rather than 12, its multiples generate *two* sequences, A and B, of recurring decimal digits:

1/13 = 0·[076923] A	7/13 = 0·[538461] B
2/13 = 0·[153846] B	8/13 = 0·[615384} B
3/13 = 0·[230769] A	9/13 = 0·[692307] A
4/13 = 0·[307692] A	10/13 = 0·[769230] A
5/13 = 0·[384615] B	11/13 = 0·[846153] B
6/13 = 0·[461538] B	12/13 = 0·[923076] A

Table 2

The A and B sequences are symmetrically placed about 7/13. There is scope to explore this topic further.

4. Generalisations of the original problem

The given question is clearly capable of being generalised; for example

- a_k becomes other than 1
- the multiplicative factor, *t*, becomes other than 3
- the base for representing the integers becomes other than 10.

Carrying through the earlier method of solution of equations (1) and (2) for the case $a_k = 2$ and $t = 3$ yields the solution N = 285714, 3N = 857142. Reference to Table 1 shows that N is the integer part of $2/7 \times 10^6$. This table also shows that a solution is only possible for $a_k = 1$ or 2 since any larger value of a_k would give 3N exceeding 7/7 and thus introduce additional digits into N and 3N which are not in the cycle (142857).

To illustrate this last point consider the case $a_k = 4$ with $t = 3$.

Case a_k = 4

4 is in the group (142857) but nevertheless $a_k = 4$ also fails to give a solution:

p = 1: $3a_0 \equiv 4 \mod 10$ so $a_0 = 8$

p = 2: $30a_1 \equiv 7a_0 + 4 = 60 \quad \mod 100$

$a_1 \equiv 2 \mod 10$ so $a_1 = 2$.

p = 3: $300a_2 \equiv 70a_1 + 7a_0 + 3 = 200 \mod 10^3$

$3a_2 \equiv 2 \mod 10$ so $a_2 = 4$.

We have arrived at $a_j = a_k$ and generated N = 428, but 3 x 428 = 1284, not 284. This corresponds to 4/7 x 3 = 12/7 = 1·$\underline{714285}$ where the leading (integer part) 1 is an additional digit not in the cycle.

Variants of the question posed, however, are readily generated. For example, we see that a multiplicative factor *t* of 2 yields solution pairs derived from the pairs of fractions (1/7, 2/7), (2/7, 4/7) and (3/7, 6/7) if in each case the first *two* digits are moved to the end of the numeral. For example, 285714 x 2 = 571428. Similarly a factor of *t* = 5 gives a solution to the reverse permutation of moving the rightmost digit to the leading position, *viz.* 142857 x 5 = 714285.

Prime numbers other than 7 furnish sequence of digits from the recurring part of their decimal representation which can provide similar cyclic properties on multiplication. We have multiples of 1/13 in Table 2 above, and the values for 1/19 are in Table 3 below.

1/19 = 0·[052631578947368421] *	10/19 = 0·[526315789473684210]
2/19 = 0·[105263157894736842] *	11/19 = 0·[578947368421052631]
3/19 = 0·[157894736842105263]	12/19 = 0·[631578947368421052]
4/19 = 0·[210526315789473684] *	13/19 = 0·[684210526315789473]
5/19 = 0·[263157894736842105]	14/19 = 0·[736842105263157894]
6/19 = 0·[315789473684210526]	15/19 = 0·[789473684210526315]
7/19 = 0·[368421052631578947]	16/19 = 0·[842105263157894736] *
8/19 = 0·[421052631578947368] *	17/19 = 0·[894736842105263157]
9/19 = 0·[473684210526315789]	18/19 = 0·[947368421052631578]

Table 3

Thus

- 2 x 052631578947368421 = 105263157894736842 (t = 2, last digit moved to leftmost position)
- 2 x 105263157894736842 = 210526315789473684 (t = 2, last digit moved to leftmost position)

The fractions marked * form a sequence in which doubling the integer corresponds to moving the least digit to the leading position. (Note that in the original problem the leading digit was moved to the least significant position.)

Other patterns can be picked out. For instance with t = 5 and the leading two digits moved to the two rightmost positions

- 5 x 105263157894736842 = 526315789473684210 and
- 5 x 157894736842105263 = 789473684210526315

One can thus construct many ‘problems’ similar to that given.

John Coffey