Q8 : Find an integer N with digits $abcabd$, with $d = c + 1$, such that N is a perfect square.

This was set in the British Mathematical Olympiad in January 1993 with the following wording

> "Find, showing your method, a six-digit integer N with the following properties:
> (i) N is a perfect square,
> (ii) the number formed by the last three digits of N is exactly one greater than the number formed by the first three digits of n. (Thus N might look like 123124, although this is not a square.)"

We could, of course, just write a computer program to search through all squares which are 6-digit integers, but let's try to narrow the search by some analysis. We are given that $N - 1$ has the digits $abcabc$. This means that $N - 1 = 1001 \times (abc)$. Let A be the integer with digits abc. Also N is a perfect square, k^2, say. We therefore have

$$1001A = k^2 - 1.$$

We can put some bounds on k. If $k < 317$, its square will only be a 5-digit integer, and if $k > 999$ its square will have seven digits.

$k^2 - 1$ factorises into $(k - 1)(k + 1)$. This means that $1001A$ can be expressed as the product of two factors which differ by 2. Suppose, therefore, that A is written as the product of two factors G and H; that is $A = GH$. Note also that $1001 = 7 \times 11 \times 13$. Then there are only three options for partitioning the divisors 7, 11 and 13 between the two factors $(k - 1)$ and $(k + 1)$:

$(7 \times 11G) \times (13H) = (k - 1)(k + 1)$ which implies that $77G \equiv \pm 2 \bmod 13$, or
$(7 \times 13G) \times (11H) = (k - 1)(k + 1)$ which implies that $91G \equiv \pm 2 \bmod 11$, or
$(11 \times 13G) \times (7H) = (k - 1)(k + 1)$ which implies that $143G \equiv \pm 2 \bmod 7$.

This lets us narrow the search for k, and hence for N, to multiples of 77, 91 and 143. The table below shows multiples up to 1001 and their respective values *mod* 13, 11 and 7. The multiples which are equivalent to $+2$ or -2 are marked in bold.

multiple	×77	mod13	×91	mod11	×143	mod7
× 1	77	12	91	3	143	3
× 2	**154**	**11**	182	6	286	6
× 3	231	10	**273**	**9**	**429**	**2**
× 4	308	9	364	1	**572**	**5**
× 5	385	8	455	4	715	1
× 6	462	7	546	7	858	4
× 7	539	6	637	10	1001	0
× 8	616	5	**728**	**2**	...	...
× 9	693	4	819	5	...	...
×10	770	3	910	8	...	...
×11	**847**	**2**	1001	0	....	...
×12	924	1	...	...	...	...

In the third line $429 \equiv 2 \bmod 7$, so 7 divides 427. Then $3 \times 143 = 429$ and $427 = 7 \times 61$, giving $427 \times 429 = 1001 \times 183 = 183183$. So $428^2 = 183184$, which is one solution to the question. Picking out the other integers in **bold** between 317 and 999, we find

$$573^3 = 328329$$

$$727^2 = 528529$$

$$846^2 = 715716.$$

Strictly, these are the only solutions to the question as stated, but note, in addition, that $155^2 = 024025$ and $274^2 = 075076$.

Finally, I have confirmed these results by a computer search of all squares of integers between 317 and 999 which have the pattern of digits *abcabd*. The results are 183184, 205209, 300304, 328329, 528529, 715716 and 732736. It is interesting that the only extra integers in this list have $d = c + 4$. These correspond to the relation

$$1001A = k^2 - 4 = (k-2)(k+2)$$

and hence to values $\equiv \pm 4$ in columns 3, 5 and 7 of the above table. For example, $858 = 7 \times 143 \equiv 4 \bmod 7$ so 7 divides 854. In fact $854 = 122 \times 7$. We thus find that $856^2 = 732736$.

However, not all squares found by this method have the pattern of digits *abcabd*. For example, $693 = 9 \times 77 \equiv 4 \bmod 13$ so 13 divides 689; $689 \div 13 = 53$. We arrive at $A = 53 \times 9 = 477$ and $691^2 = 477481$.

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