**Q9 : Let $a$ and $b$ be positive integers such that $ab+1$ divides $a^2+b^2$. Show that the quotient $(a^2+b^2)/(ab+1)$ is always a perfect square.**

This was set in the 29th International Mathematical Olympiad in 1988.

Suppose that $n$ is the integer quotient when $ab+1$ divides $a^2+b^2$. Then

$$a^2+b^2 = n(ab+1) = nab+n. \tag{1}$$

The question as posed asks us merely to prove that $n$ is always a perfect square. We are not asked to solve the Diophantine equation for integers $a$, $b$ and $n$, though we will do so. Denote a solution set by $\{a,b,n\}$.

**1. Elementary solutions**

Let's start by identifying the easy solutions.

1) We are given that both $a$ and $b$ are positive so $n>0$. Note that the problem is symmetric in $a$ and $b$. However we can have $a=b$ only if $a=b=1$, making $n=1=1^2$, its least non-zero integer value. For all other values of $n$, $a$ and $b$ are unequal.

2) A limiting case has $b=0$, $n=a^2$, equivalently $a=0$, $n=b^2$.

3) In Eq (1) match the terms, setting $b^2=n$ (which makes $n$ a square, as claimed) and $a^2=nab$, so that $a=b^3$. The corresponding quotient is

$$\frac{a^2+b^2}{ab+1} = \frac{b^6+b^2}{b^4+1} = b^2.$$

**2. An argument by the method of descents**

To search for further solutions, first express Eq (1) as a quadratic equation in $a$, and as an equivalent equation in $b$:

$$a^2 - nba + (b^2-n) = 0, \tag{2$a$}$$

$$b^2 - nab + (a^2-n) = 0. \tag{2$b$}$$

My first attempt at searching led from Eq (2) on a circuitous journey into Pell's equation. Though this is interesting in itself and is given in §3, I am pleased to thank a reader, Mr Jim Pearson, for outlining a more direct solution which I will present first. He uses that fact that the constant term in a quadratic equation is the product of the two roots. Call the roots of Eq (2a) $\alpha_1$ and $\alpha_2$ and suppose that $\alpha_1>\alpha_2$ (since by §1 they cannot be equal for general $n$). Thus

$$(a-\alpha_1)(a-\alpha_2) = a^2 - nba + (b^2-n) \quad \text{so} \quad \alpha_1\alpha_2 = b^2-n \tag{3}.$$

Accept for the moment that $n>0$ and $n<b^2$ so $\alpha_1\alpha_2 > b^2 > b^2-n$. Since $a \neq b$, we can take $\alpha_1 > b$, $\alpha_2 < b$. So $\{\alpha_1, b, n\}$ form one solution set, and $\{b, \alpha_2, n\}$ another.

The equivalent equation Eq (2b) has roots $\beta_1$, $\beta_2$. If $\beta_1 > a$, $\beta_2 < a$. But the roots $\beta_1$, $\beta_2$ from Eq (2b) must be $b$ values which feature in Eq (2a). Similarly $\alpha_1$, $\alpha_2$ from Eq (2a) must be $a$ values in Eq (2b).

So we generate a sequence of solution sets $\{\alpha_1, \beta_1, n\}$, $\{\beta_1, \alpha_2, n\}$, $\{\alpha_2, \beta_2, n\}$, $\{\beta_2, \alpha_3, n\}$, etc. with $n$ remaining the same. Clearly this descent cannot go on indefinitely. Suppose the least root of all is $u_0$; that is, the sequence is $\alpha_1 > \beta_1 > \alpha_2 > \beta_2 > \alpha_3 > \beta_3 > \dots u_0$. Then one of the following must be true : i) $u_0 > 0$, ii) $u_0 < 0$, iii) $u_0 = 0$. Examine these in turn.

Case i ) The sequence cannot stop at some $u_0 > 0$ because for every $u_0 > 0$ there is the solution set $\{u_0, 0, u_0^2\}$ containing 0.

Case ii ) Conjecture a solution set $\{u_1, u_0, \nu\}$ with $u_1 > 0$, $u_0 < 0$. Then the denominator in the original problem is $u_1 u_0 + 1$. We cannot have $u_1 = 1$, $u_0 = -1$ since that would give division by zero. So $u_1 u_0 + 1$ must be negative. Since $u_1^2 + u_0^2 > 0$ always, we conclude that $\nu < 0$. Clearly this $\nu$ cannot be identified with $n$ in a descending sequence of solutions $\{a, b, n\}$ because, as we observed above, $n > 0$ and $n$ is *constant* in any sequence.

Case iii) We conclude that $u_0 = 0$.

From the elementary solution set $\{u_1, 0, u_1^2\}$ and Eq (2a), (2b) we retrace our steps up the sequence of descent to build a complete set of solutions starting from any chosen value of $u_1$. The sequence starts

$$0,, \quad u_1, \quad u_1^3, \quad u_1(u_1^4 - 1), \quad u_1^3(u_1^4 - 2), \quad u_1(u_1^8 - 3u_1^4 + 1) \quad \dots$$

At this stage I would like to introduce a new notation in $u$ for this sequence:

$$u_0 = 0, \;\; u_1, \;\; u_2 = u_1^3, \;\; u_3, \; .... \; u_k, \; ....$$

$a$ and $b$ in the original problem can be any adjacent pair in this sequence $\{u_k\}$.

The terms satisfy the recursion relation in Eq (3), namely $u_{k+1} u_{k-1} = u_k^2 - n$ with $n = u_1^2$. This can be expressed in the alternative form

$$u_{k+1} = n u_k - u_{k-1}. \tag{4}$$

To see this use the definition of $n$ in the original problem:

$$n = \frac{u_k^2 + u_{k-1}^2}{u_k u_{k-1} + 1}.$$

This can be manipulated into

$$u_{k+1} u_{k-1} + n + u_{k-1}^2 = u_k^2 + u_{k-1}^2 = n u_k u_{k-1} + n$$

from which Eq (4) follows immediately. The auxiliary equation for recursion relation (4) is

$$U^2 - nU + 1 = 0 \quad \text{which has roots} \quad \tfrac{1}{2}[n \pm \sqrt{n^2 - 4}\,].$$

These zeros correspond to the limit for the ratio of adjacent terms in the sequence (one being the reciprocal of the other).

## 3. Relation to Pell's equation

Return to the quadratic Eq (2a) and solve it by the familiar formula:

$$a = \tfrac{1}{2}(nb \pm \sqrt{(nb)^2 - 4(b^2 - n)}\,).$$

The square root must evaluate to an integer, $r$, say:

$$(nb)^2 - 4(b^2 - n) = r^2 \tag{5}$$

$$\text{so that} \quad a = \tfrac{1}{2}(nb \pm r).$$

Note in passing that $r$ will be odd or even just as $nb$ is odd or even. Elementary solution number 2 of §1 corresponds to $b^2 - n = 0$.

At this stage I am looking for analogous quadratic equations in integers for which solutions are known. There is the well known Pell's equation, so we can enquire whether Eq (3) might be an instance. Pell's equation is

$$x^2 - Ny^2 = \pm K$$

where $x$, $y$, $N$ and $K$ are integers and $N$ is not a perfect square. In the 'purest' form $K = 1$. It is well known that the solutions to Pell's equation are related to the continued fraction expansion of $\sqrt{N}$. One evaluates successive convergents $p_j/q_j$ of this continued fraction and equates the numerator $p_j$ to $x$ and the denominator $q_j$ to $y$. I have written my own account of this interesting topic in an article on my web site at *www.mathstudio.co.uk*.

Recasting Eq (5) we obtain

$$\frac{r^2}{4n} - (n^2 - 4)\frac{b^2}{4n} = 1.$$

If we suppose $n$ to be a perfect square, $n = m^2$, $m$ an integer,

$$x^2 = \frac{r^2}{4m^2} \quad \text{and} \quad y^2 = \frac{b^2}{4m^2}\ ,$$

then we have Pell's equation with $N = m^4 - 4$ and $K = 1$:

$$\left(\frac{r}{2m}\right)^2 - (m^4 - 4)\left(\frac{b}{2m}\right)^2 = 1. \tag{6$a$}$$

If we allow $K$ to be 4, we have the more general form

$$\left(\frac{r}{m}\right)^2 - (m^4 - 4)\left(\frac{b}{m}\right)^2 = 4. \tag{6$b$}$$

and if $K$ is $m^2$, we have another form

$$\left(\frac{r}{2}\right)^2 - (m^4 - 4)\left(\frac{b}{2}\right)^2 = m^2. \tag{6$c$}$$

Each of these can in principle furnish solutions based on the continued fraction expansion of $\sqrt{m^4 - 4}$. Before looking further into the algebra, consider some numerical examples.

**Case** $m = 2$**,** $n = 4$
The lowest meaningful value for $m$ is 2. Eq (6a) becomes

$$\left(\frac{r}{4}\right)^2 - 3\left(\frac{b}{2}\right)^2 = 1. \tag{7}$$

The continued fraction expansion of $\sqrt{3}$ is $\{1 : 1, 2, 1, 2, 1, 2, 1, \ldots\}$ or $\{1 : \underline{1, 2}\}$ where the underlined sequence of partial quotients recurs. Successive convergents $p_j/q_j$ of this are 2/1, 5/3, 7/4, 19/11, 26/15, 71/41, 97/56, 265/153, 362/209, 989/571, 1351/780, etc. (These are formed by truncating the continued fraction and evaluating it as an ordinary fraction.) Index these convergents with $j$, starting at $j = 1$. The numerators and denominators satisfy the recursion relations

$$p_j = p_{j-1} + p_{j-2} \quad , \qquad q_{j+1} = q_{j-1} + q_{j-2} \quad , \qquad j \text{ odd}, \tag{8a}$$

$$p_j = 2p_{j-1} + p_{j-2} \quad , \qquad q_{j+1} = 2q_{j-1} + q_{j-2} \quad , \qquad j \text{ even}. \tag{8b}$$

Here the 1 and 2 coefficients arise from the sequence of partial quotients in the expansion of $\sqrt{3}$.

We are going to calculate $p_j^2 - 3q_j^2$ and compare it with Eq (7). Taking the convergents in turn:

$$2/1: \quad 2^2 - 3 \times 1^2 = 1 \text{ so } b = 2, \ r = 8 \text{ and } 2a = 8 \pm 8 \text{ so } \{a, b, n\} = \{8, 2, 4\} \text{ or } \{0, 2, 4\}.$$

Both these are elementary solution sets in §1 for $n = 4$.

$$5/3: \quad 5^2 - 3 \times 3^2 = -2 \text{ so no solution.}$$

$$7/4: \quad 7^2 - 3 \times 4^2 = 1 \text{ so } b = 8, \ r = 28 \text{ and } 2a = 32 \pm 28$$

so $\{2, 8, 4\}$ is one solution set and $\{30, 8, 4\}$ another, consistent with our findings in §2. Note how the solution $\{8, 2, 4\} \equiv \{2, 8, 4\}$ has been derived from two distinct convergents of $\sqrt{3}$.

$$19/11: \quad 19^2 - 3 \times 11^2 = -2 \text{ so no solution.}$$

$$26/15: \quad 26^2 - 3 \times 15^2 = 1 \text{ so } b = 30, \ r = 104 \text{ and } 2a = 120 \pm 104$$

so solutions are $\{8, 30, 4\}$ again and $\{112, 30, 4\}$. It becomes clear that each odd indexed convergent gives two solutions, one of which repeats one of the previous pair though with $a$ and $b$ swapped over. We can therefore express this infinite set of solutions, for $n = 4$, as a sequence of integers $u_k$ such that any adjacent pair may to taken as $a$ or $b$:

$$2, \ 8, \ 30, \ 112, \ 418, \ 1560, \ 5822, \ 21728, \ 81090, \ldots$$

The terms satisfy the two recursion relations $u_{k+1} = 2u_k + \sqrt{3u_k^2 + 4}$ and $u_{k+1} = 4u_k - u_{k-1}$ (with $u_0 = 0$, $u_1 = 2$). The ratio of adjacent terms tends to the limit $2 + \sqrt{3}$, consistent with the value given at the end of §2. We will delay proving these results until §4.

**Case** $m = 3$**,** $n = 9$
Eq (6a) becomes

$$\left(\frac{r}{6}\right)^2 - 77\left(\frac{b}{6}\right)^2 = 1.$$

while Eq (6b) becomes

$$\left(\frac{r}{3}\right)^2 - 77\left(\frac{b}{3}\right)^2 = 4.$$

The continued fraction expansion of $\sqrt{77}$ is $\{8 : \underline{1, 3, 2, 3, 1, 16}\}$, with successive convergents and associated Pell equations as follows:

$$\begin{aligned} 9/1: &\quad 9^2 - 77 \times 1^2 = 4 \\ 35/4: &\quad 35^2 - 77 \times 4^2 = -7 \\ 79/9: &\quad 79^2 - 77 \times 9^2 = 4 \\ 272/31: &\quad 272^2 - 77 \times 31^2 = -13 \\ 351/40: &\quad 351^2 - 77 \times 40^2 = 1 \end{aligned}$$

Alternate convergents give differences which alternate in sign (a characteristic of continued fractions). Only the positive ones are of use to us (see footnote). Some other useful convergents include

$$\begin{aligned} 6239/711: &\quad 6239^2 - 77 \times 711^2 = 4 \\ 55449/6319: &\quad 55449^2 - 77 \times 6319^2 = 4 \\ 246401/28080: &\quad 246401^2 - 77 \times 28080^2 = 1 \end{aligned}$$

As with the case $m = 2$, we can build a sequence of integers $u_k$ such that any adjacent pair gives $a$ or $b$ satisfying the original equation, Eq (1):

$$3,\ 27,\ 240,\ 2133,\ 18957,\ 168480, \ldots$$

The difference equation for terms is $u_{k+1} = 9u_k - u_{k-1}$, with $u_0 = 0$, $u_1 = 3$. The limiting ratio of adjacent terms is $(9 + \sqrt{77})/2$, which is one root of the related quadratic equation $u^2 - 9u + 1 = 0$. (The other root is the reciprocal ratio.)

**Cases $m = 4$, $n = 16$ and $m = 5$, $n = 25$**

We could continue evaluating with $m = 4$, etc., but perhaps we might anticipate that solutions $a$ and $b$ will be adjacent pairs from sequences of the form $u_{k+1} = m^2 u_k - u_{k-1}$, $u_0 = 0$, $u_1 = m$. To try this, the sequences for $m = 4$ and $m = 5$ are

$$4,\ 64,\ 1020,\ 16256,\ 259076, \ldots$$

$$5,\ 125,\ 3120,\ 77875,\ 1943755, \ldots$$

As a numerical test, $(1943755^2 + 77875^2)/(1943755 \times 77875 + 1) = 3784248015650/151369920626 = 25$, so this looks promising.

## 4. Some algebra relating to §3

In this section we will tie in the discussion of Pell's equation in §3 with the method of descents in §2.

Let us derive the general case of the two recursion relations already noted for the case $m = 2$, namely

$$u_{k+1} = 2u_k + \sqrt{3u_k^2 + 4} \quad \text{and} \quad u_{k+1} = 4u_k - u_{k-1} \quad \text{with } u_0 = 0, u_1 = 2.$$

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The right side being 1 or 4 in not very significant. It arises because in some convergents the numerator and denominator have a common factor which has been cancelled.

Go back to Eqs (5) and (6a), and match (6a) to $p_j^2 - (n^2-4)q_j^2 = 1$. We thus identify $b$ with $2mq_j$ and $r$ with $2mp_j$. Since $b$ and $a$ are consecutives in the series $u_k$, let $b = u_k$ and $a = u_{k+1} = \frac{1}{2}(nb+r)$. (Note that in general $k \neq j$ because only convergents with $j$ odd give a solution to Pell's equation Eq (4).) Hence

$$u_{k+1} = \tfrac{1}{2}(nu_k + 2mp_j) = \tfrac{1}{2}(nu_k + 2m\sqrt{1+(n^2-4)q_j^2} = \tfrac{1}{2}(nu_k + \sqrt{4n+(n^2-4)u_k^2}$$

since $n = m^2$. This establishes the generality of one relation.

The other can be obtained from it. Decrease the index by 1 and rearrange to get

$$(2u_k - nu_{k-1})^2 = 4n + (n^2-4)u_{k-1}^2,$$

$$\text{which is } \; u_k^2 - nu_ku_{k-1} + u_{k-1}^2 = n.$$

Replace the first $n$ in $\sqrt{4n+(n^2-4)u_k^2}$ with the above, note that $n^2u_k^2 - 4nu_ku_{k-1} + 4u_{k-1}^2$ is a perfect square, and the result $u_{k+1} = nu_k - u_{k-1}$ follows immediately.

Now a digression into the form of the continued fractions for $\sqrt{N}$ where $N = (m^2-4)$. If $m$ is odd these continued fractions can be quite complicated; for example if $m = 3$, $\sqrt{77} = \{8 : \underline{1,3,2,3,1,16}\}$. However if $m$ is even, they have the fairly simple form $\{c : \underline{1, 2c}\}$. Table 1 gives some illustrative values. I have written $N = m^4 - 4$, $m = 2\mu$, $n = 4\mu^2$.

**Table 1**

Continued fraction representation of $\sqrt{N}$ for $N = 4\mu^4 - 1$

| $\mu$ | $n$ | $4\mu^4-1$ | fraction | $c$ |
|---|---|---|---|---|
| 1 | 4 | 3 | $\{1 : \underline{1,2}\}$ | 1 |
| 2 | 16 | 63 | $\{7 : \underline{1,14}\}$ | 7 |
| 3 | 36 | 323 | $\{17 : \underline{1,34}\}$ | 17 |
| 4 | 64 | 1023 | $\{31 : \underline{1,62}\}$ | 31 |
| 5 | 100 | 2499 | $\{49 : \underline{1,98}\}$ | 49 |
| 6 | 144 | 5183 | $\{71 : \underline{1,142}\}$ | 71 |
| 7 | 196 | 9603 | $\{97 : \underline{1,194}\}$ | 97 |

In general $c = 2\mu^2 - 1$.

To understand this pattern, observe that $\sqrt{N} = \{c : \underline{1, 2c}\}$ means

$$\sqrt{N} = c + \cfrac{1}{1 + \cfrac{1}{2c + \cfrac{1}{1 + \cfrac{1}{2c + \ldots +}}}}.$$

Let

$$\theta = 2c + \cfrac{1}{1 + \cfrac{1}{2c + \ldots}}$$

so $\sqrt{N} = \theta - c$ and

$$\theta = 2c + \frac{1}{1 + \frac{1}{\theta}}.$$

This gives a quadratic equation in $\theta$ whose positive root is $\theta = c + \sqrt{c^2 + 2c}$, making $N = c^2 + 2c$, consistent with Table 1. Substituting $c = 2\mu^2 - 1$ recovers $N = 4\mu^4 - 1$.

Still working with $m$ even, note also that the numerators and denominators of the convergents of $\{c : \underline{1, 2c}\}$ satisfy recursion relations (equivalent to Eq (8)):

$$p_j = p_{j-1} + p_{j-2} \ , \qquad q_{j+1} = q_{j-1} + q_{j-2} \ , \qquad j \text{ odd}, \tag{9a}$$

$$p_j = 2cp_{j-1} + p_{j-2} \ , \qquad q_{j+1} = 2cq_{j-1} + q_{j-2} \ , \qquad j \text{ even}. \tag{9b}$$

with $p_0 = c$, $q_0 = 1$, $p_1 = c + 1$, $q_1 = 1$. Evaluating we find that

$$p_2 = c(2c + 3), \qquad q_2 = 2c + 1$$

$$p_3 = 2c^2 + 4c + 1, \qquad q_2 = 2(c + 1), \text{ etc.}$$

Referring back to Eq (6a), if we solve $p_j^2 - Nq_j^2 = 1$ we find solutions for all odd $j$ at $N = c(c+2)$. Values for $c = 1, 2, 3, \ldots$ are 3, 8, 15, 24, 35, 48, 63, 80, 99, etc. Of these 3 and 63 feature in Table 1 at $n = 4$ and 16 respectively.

I leave further investigation of the case $m$ odd to another interested reader.

Note in closing that there are solutions with $n$ negative, though $-n$ is not a perfect square. For example, $a = 1$, $b = -3$, $n = -5$. These obey the same recursion relation as before, namely $u_{k+1} = nu_k - u_{k-1}$ where $a$ and $b$ are any adjacent pair of numbers in the sequence $\{u_k\}$. The example just given continues

$$1\ , \quad -3\ , \quad 14\ , \quad -67\ , \quad 321, \ldots$$

## 5. Summary

We have found all sets of solutions $\{a, b, n = m^2\}$, indexed by constant $m$, each of which has an infinite number of solutions according to the position of $a$ and $b$ along the sequence $u_k$. Two recursion relations are

$$u_{k+1} = \tfrac{1}{2}(nu_k + \sqrt{4n + (n^2 - 4)u_k^2}\ ),$$

$$u_{k+1} = nu_k - u_{k-1}, \text{ starting from } u_0 = 0, u_1 = m.$$

We have also found the limiting ratio $a/b$ as $a > b \to \infty$ to be $\frac{1}{2}[m^2 + \sqrt{m^4 - 4}\ ]$. All solution sets are determined by the convergents of the continued fraction expansion of $\sqrt{m^4 - 4}$. If $m$ is even, the continued fraction has the fairly simple form $\{c : \underline{1, 2c}\}$. In all cases $n$ is a perfect square because $n$ is constant for any series of solutions, and the series starts at $\{m, 0, m^2\}$.

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Text revised in light of Jim Pearson's comments, Jan 2010.