# Fitting an Ellipse inside a Triangle and a Pentagon

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**The Question:** Given the co-ordinates of the five vertices of an arbitrary convex pentagon, construct the unique ellipse inside the pentagon such that each of its five sides is a tangent to the ellipse. As a simpler challenge, construct the inscribed circle of a given triangle.

**Comment:** A circle has equation $(x - x_0)^2 + (y - y_0)^2 = r^2$ in Cartesian co-ordinates and so has three independent parameters: $x_0$, $y_0$ and $r$. The three sides of a triangle provide sufficient information for three independent simultaneous equations to be constructed whose solutions are $x_0$, $y_0$ and $r$. A general ellipse has the more complicated equation

$$\frac{[(x - x_0)\cos\theta + (y - y_0)\sin\theta]^2}{p^2} + \frac{[-(x - x_0)\sin\theta + (y - y_0)\cos\theta]^2}{q^2} = 1. \tag{1}$$

This has five parameters: the two off-sets $x_0$, $y_0$, the semi-axes $p$ and $q$, and the orientation of the axes $\theta$. This suggests that there is a unique maximum ellipse which can be drawn within and touching any convex pentagon.

This raises the further question of what smooth closed figure, similar to an indented ellipse, can be inscribed within a concave pentagon.

## 1 Circle inside a triangle

### 1.1 Euclidean geometry

The solution to the simpler question of the circle inscribed within a triangle was known to Euclid and indeed is almost obvious. As Figure 1 shows, the centre of the inscribed circle must lie on the bisectors of the three angles. Where the radii (dashed red lines) meet the sides, they form three pairs of isosceles triangles. With some trigonometry the radius $r$ can be found. Let $h$ be the distance from vertex A to where the radii intersect sides $b$ and $c$. Let $k$ be the equivalent distance from C. Then $l = b - h$ and, around vertex B, $l = a - k = c - h$ so $h = (-a + b + c)/2$, $k = (a - b + c)/2$, $l = (a + b - c)/2$. The half angles at the vertices are marked as $\alpha$, $\beta$, $\gamma$ in Figure 1. The angle A has cosine given by the familiar formula

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} \quad \text{from which} \quad \cos\alpha = \sqrt{\frac{\cos A + 1}{2}} = \sqrt{\frac{-a^2 + (b + c)^2}{4bc}}.$$

We now find $\tan(A/2) = \tan\alpha$ from

$$\tan^2\alpha + 1 = \frac{1}{\cos^2\alpha} \quad \text{from which} \quad \tan\alpha = \sqrt{\frac{a^2 - (b - c)^2}{-a^2 + (b + c)^2}}.$$

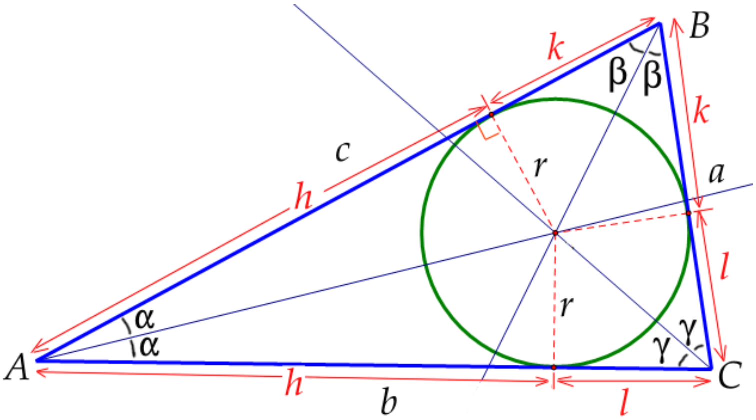

Figure 1: The centre of the inscribed circle of a triangle is where the angle bisectors meet.

Finally

$$r = h\tan\alpha = \tfrac{1}{2}(-a+b+c)\sqrt{\frac{a^2-(b-c)^2}{-a^2+(b+c)^2}} = \sqrt{\frac{(-a+b+c)(a-b+c)(a+b-c)}{4(a+b+c)}}.$$

This has the remarkably simple and symmetric form

$$r^2 = \frac{hkl}{s} = \frac{(s-a)(s-b)(s-c)}{s}, \qquad s = \tfrac{1}{2}(a+b+c), \text{ the semi-perimeter.} \tag{2}$$

Note that there is a close relation with the area of the triangle: area $= rs$, a remarkable property of the inscribed circle.

That is by no means the end of the story. The obtuse angles at the vertices A, B, C can also be bisected and their bisectors are concurrent at three points exterior to the triangle. These are the centres of three circles labelled 2, 3, 4 in Figure 2, unfortunately a rather involved diagram. By examining the pairs of isosceles triangles it is straightforward to show that the coloured sets of line sectors in the figure have the same length according to colour. These lengths were introduced above: $h = s-a$ (red), $k = s-b$ (green), $l = s-c$ (blue). For example, AP has length $k$ and so the distance CP is $b+k = s$, the semi-perimeter. Similarly BQ is $h$ and CQ is $h+a = s$. In fact all similar distances from a vertex to a tangent point on the opposite circle are $s$. It follows that in circle 4, for instance, with radius $r_4$

$$r_4 = s\tan\gamma \quad \text{and similarly} \quad r_2 = s\tan\alpha, \quad r_3 = s\tan\beta \tag{3}$$

$$\text{where} \quad \tan\beta = \sqrt{\frac{b^2-(c-a)^2}{-b^2+(c+a)^2}}, \qquad \tan\gamma = \sqrt{\frac{c^2-(a-b)^2}{-c^2+(a+b)^2}}.$$

We arrive at these beautifully simple and symmetric expressions for the radii of the four circles:

$$r_1 = \sqrt{\frac{hkl}{s}}, \quad r_2 = \sqrt{\frac{kls}{h}}, \quad r_3 = \sqrt{\frac{hls}{k}}, \quad r_4 = \sqrt{\frac{hks}{l}}. \tag{4}$$

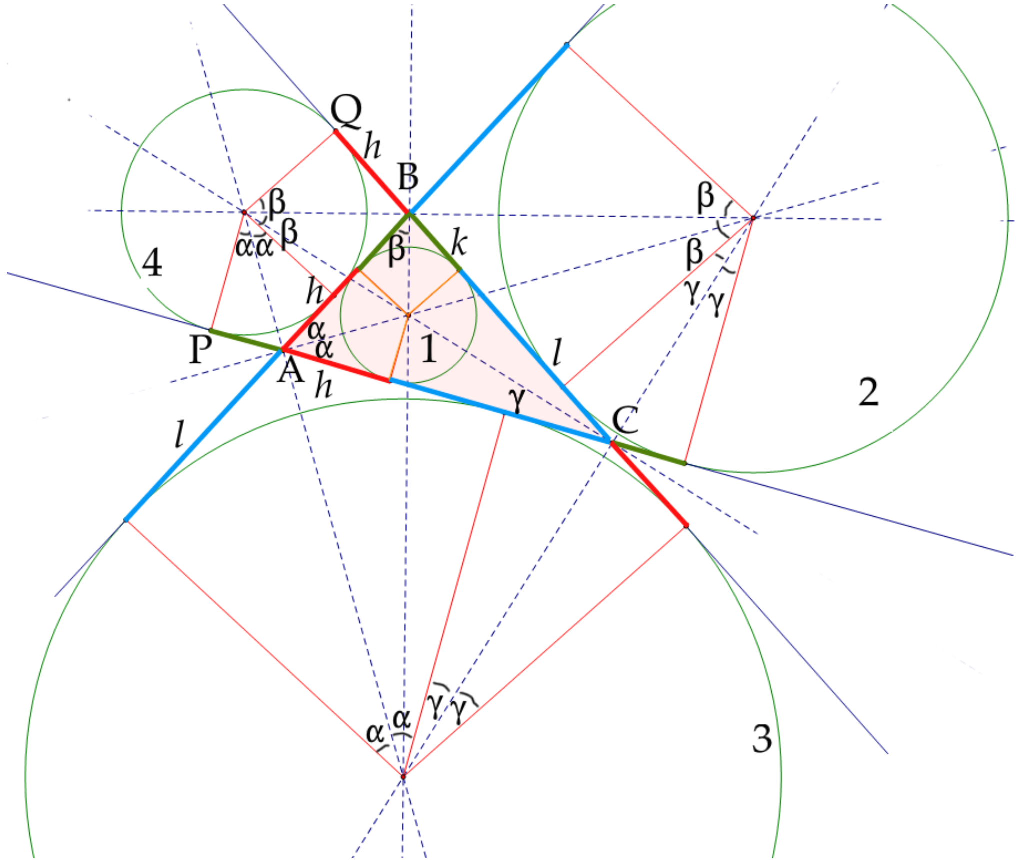


Figure 2: Four circles whose centres are defined by the 3-way intersection of the bisectors, both acute and obtuse, of the three vertex angles. The bisectors are shown dotted. Red length $=h$, green $= k$, blue $= l$.

To complete this geometric story look at the triangle formed by the centres of the three exterior tangent circles. The notation shows that its internal angles are $\alpha+\beta$, $\beta+\gamma$ and $\gamma+\alpha$. The length of its left side passing through A is

$$\sqrt{r_4^2+k^2}+\sqrt{r_3^2+l^2} = \sqrt{\frac{bck}{l}}+\sqrt{\frac{bcl}{k}} = 2a\sqrt{\frac{bc}{a^2-(b-c)^2}}. \qquad 5)$$

and similarly for the other two sides.

### 1.2 Analytic geometry of the triangle

Now we examine the same problem using co-ordinate geometry. This will be how the pentagon problem is approached in §3. I propose to look at this in two ways.

**Approach 1:** We will use the knowledge that, because all radii of a circle have the same length, the centre of the inscribed circle must lie at the intersection of the angle bisectors. Let the three vertices have co-ordinates A:$(x_A, y_A)$, B:$(x_B, y_B)$, C:$(x_C, y_C)$. If $(x, y)$ is a general point on side $c =$ AB, its equation is

$$\text{Side } c = \text{AB:} \quad y = \frac{y_B - y_A}{x_B - x_A}(x - x_A) + y_A = m_c(x - x_A) + y_A \qquad (3)$$

and similarly for the other two sides. The notation can become dense and opaque so label

$$x_B - x_A = c_x,\quad y_B - y_A = c_y,\quad x_C - x_B = a_x,\quad y_C - y_B = a_y,\quad x_A - x_C = b_x,\quad y_A - y_C = b_y,$$

and the gradients of the sides as $m_a = a_y/a_x$, $m_b = b_y/b_x$, $m_c = c_y/c_x$, taking the sides in cyclic order and matching the notation somewhat to Figures 1 and 2. The inscribed circle with centre O: $(x_0, y_0)$ is $(x - x_0)^2 + (y - y_0)^2 = r^2$.

The angle at A, equal to $2\alpha$, can be bisected as follows. Using the trigonometric formula for the difference of two tangents it can fairly readily be shown that the line through vertex A which bisects its two adjacent sides with gradients $m_b$, $m_c$ respectively has gradient

$$m_\alpha = \frac{m_b m_c - 1 \pm \sqrt{(m_b^2+1)(m_c^2+1)}}{m_b + m_c} \tag{4}$$

$$\text{and equation} \quad y = m_\alpha(x - x_A) + y_A.$$

(I am a stretching the notation here by labelling the gradients of the bisectors with their semi-angle.) Solving simultaneously with the bisector of angle B = $2\beta$ gives

$$x_0 = \frac{m_\alpha x_A - y_A - m_\beta x_B + y_B}{m_\alpha - m_\beta}. \tag{5}$$

The ± sign at Eq 3 arises because there are two bisectors where two lines cross, one of the acute angle, the other the obtuse. There are four solutions for $x_0$ from Eq 4 because there are ± signs for both $m_\alpha$ and $m_\beta$, and as we have seen, theses correspond to the inscribed and three exterior circles in Figure 2. Since the second bisectors are related to $m_\alpha$ by $-1/m_\alpha$, $m_\beta$ by $-1/m_\beta$, all four $x_0$ can be readily calculated. Moreover, replacing either $m_\alpha$ or $m_\beta$ by $m_\gamma$ will give the same four $x_0$.

The formulae Eqs 3 and 4 may be straightforward to computer numerically, but do not afford the insight of the Euclidean approach and especially of Eq 4. Can the $x_0$ be expressed more transparently in terms of the side lengths and angles of the triangle? This seems unlikely because co-ordinate geometry deals in gradients and not angles, and in components of lengths projected onto the axes rather than in lengths directly.

**Approach 2:** This approach assumes no prior knowledge that the centre of the inscribed circle lies on the vertex bisectors. Instead it uses the fact that any straight line which is tangent to a curve must both meet it at a single point and have the same gradient there.

The equation of side AB, labelled $c$, is given by Eq 3 and the circle is $(x-x_0)^2+(y-y_0)^2 = r^2$. Substituting for $y$ gives a quadratic in $x$ for the points of intersection. Labelling the quadratic as

$$\alpha_c x^2 - 2\beta_c x + \gamma_c, \qquad \text{the coefficients are}$$

$$\alpha_c = 1 + m_c^2,\quad \beta_c = x_0 + m_c(m_c x_A + y_0 - y_A),\quad \gamma_c = x_0^2 + (m_c x_A + y_0 - y_A)^2 - r^2.$$

Since each side just touches the circle as a tangent, the discriminant $\Delta_c = \beta_c^2 - \alpha_c\gamma_c$ is zero:

$$(1 + m_c^2)r^2 - [m_c(x_0 - x_A) - y_0 + y_A]^2 = 0. \qquad 5c)$$

There are equivalent equations for the other two sides, giving three simultaneous equations in $x_0$, $y_0$ and $r^2$, The other two are

$$\alpha_b x^2 - 2\beta_b x + \gamma_b, \quad \alpha_b = 1 + m_b^2, \quad \beta_b = x_0 + m_b(m_b x_B + y_0 - y_B), \quad \gamma_b = x_0^2 + (m_b x_B + y_0 - y_B)^2 - r^2,$$

$$\alpha_a x^2 - 2\beta_a x + \gamma_a, \quad \alpha_a = 1 + m_a^2, \quad \beta_b = x_0 + m_a(m_a x_C + y_0 - y_C), \quad \gamma_a = x_0^2 + (m_a x_C + y_0 - y_C)^2 - r^2.$$

$$(1 + m_b^2)r^2 - [m_b(x_0 - x_B) - y_0 + y_B]^2 = 0. \tag{5b}$$

$$(1 + m_a^2)r^2 - [m_a(x_0 - x_C) - y_0 + y_C]^2 = 0. \tag{5a}$$

$r^2$ is dealt with immediately from Eq 3c leaving equations in $x_0$, $y_0$ which, being non-linear, are awkward to solve.

$$r^2 = \frac{[m_c(x_0 - x_A) - y_0 + y_A]^2}{1 + m_c^2}$$

so substitute this into Eq 5b to produce a quadratic in $x_0$ and $y_0$. If we treat this as a quadratic only in $y_0$, it can be solved for $x_0$. The resulting equation is a very messy, complicated quartic equation. Its four solutions give for $x_0$ values. If these are all real, they are the $x$ positions of the centres of the four circles which satisfy the specification of having the three sides of the triangle as tangents. This is illustrated in Figure 2. Circle 1 – defined by one of the $x_0$ roots and hence $y_0$ and $r^2$ – is the required inscribed circle. The other three sets of values give circles outside the triangle where the two of the tangents are the sides produced. The values for this example are listed in Table 1.

| Vertex | $x$ | $y$ | Side | length | gradient | Circle | $x_0$ | $y_0$ | $r$ |
|---|---|---|---|---|---|---|---|---|---|
| A | 0 | 1 | AB = $c$ | 3·606 | 3/2 | 1 | 2·20 | 1·95 | 1·30 |
| B | 2 | 4 | BC = $a$ | 5·657 | −1 | 2 | −1·16 | 3·69 | 2·46 |
| C | 6 | 0 | CA = $b$ | 6·083 | −1/6 | 3 | 8·38 | 4·63 | 4·96 |
| | | | | | | 4 | 2·98 | −5·87 | 6·29 |

Table 1: Values used in Figure 3.

It is worth reflecting on the contrast between the Euclidean and coordinate geometry. The classical method is direct, simple, elegant and quite beautiful. It gives not only the inscribed circle, but mby extension of the sides, the three external circles also. The coordinate geometry is algebraically messy, the solution almost unusable except when numerical solutions are required to specific triangles. The Euclidean method gives an answer of simple form because it deals only with the lengths of the sides, which are the intrinsic, natural properties of the figure. Coordinate geometry here drowns in its welter of coordinates. Length in coordinate geometry often requires a square root since Pythagoras theorem has been applied. The coordinates are not intrinsic to the figures, but belong to the grid over which the triangle and circles are laid. They place the triangle and circles in a frame of reference which lies beyond the figures themselves and we are obliged to carry this frame with us throughout the calculation and into the solution.

Before moving to address the pentagon, it is wroth mentioning are two other intriguing constructions with triangles in which lines meet at a point. First, the three lines, one through each vertex, which are perpendicular to the opposite sides all meet at the one point. Second, a circle called the circumcircle, passes through the three vertices. Its centre lies on the perpendicular bisectors of the three sides.

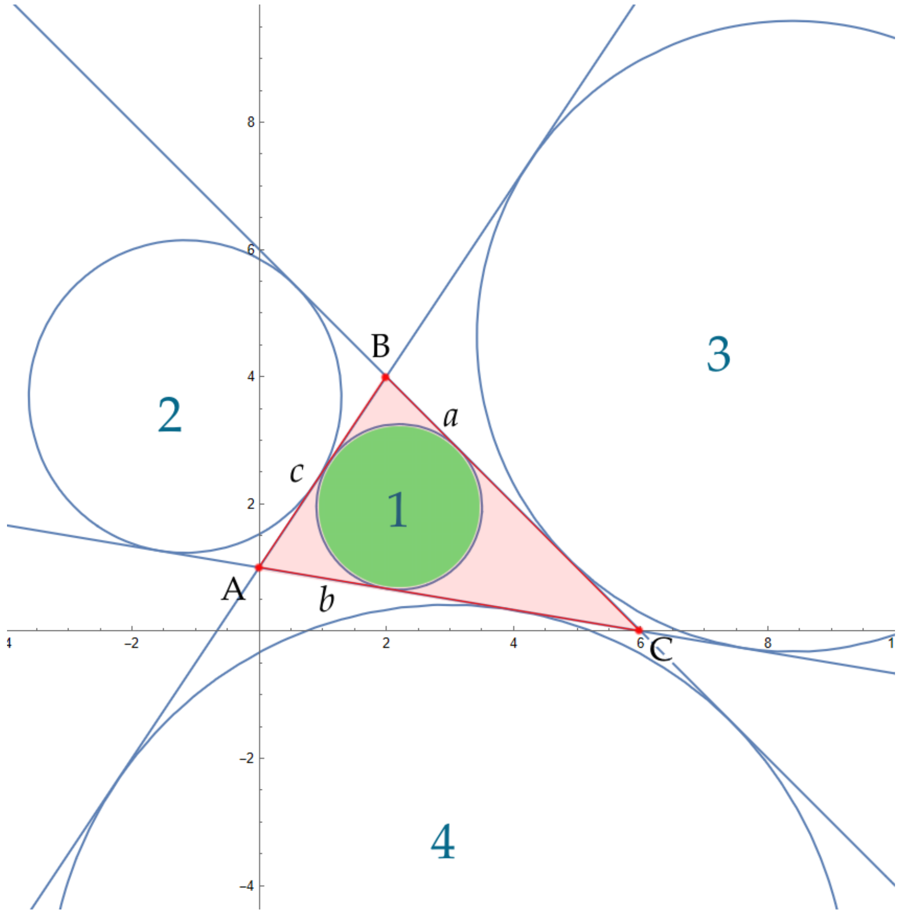


Figure 3: Inscribed and 3 associated external circles to triangle ABC using values in Table 1.

## 2 Ellipse touching a convex pentagon

The solution we seek is illustrated in Figure 4. Clearly a circle cannot touch all five sides of a general pentagon, so we look for one or more ellipses which do. I know no special property of either ellipse or pentagon which allow us to cut corners and construct the foci of these ellipses. The approach, therefore, will be to require each straight side to intersect the ellipse at one point only, where the two curves have the same gradient. This should generate five simultaneous equations. Another property which might be used is that the sum of the distances from any point on the ellipse to the two foci is constant, equal to the major axis. We shall see.

The equation of the ellipse is in Eq 1:

$$\frac{[(x-x_0)\cos\theta+(y-y_0)\sin\theta]^2}{p^2}+\frac{[-(x-x_0)\sin\theta+(y-y_0)\cos\theta]^2}{q^2} = 1. \qquad (1, copy)$$

The two foci are distance $f$ from the centre. Label the vertices of the ellipse A to E and their opposite sides $a$ to $e$ as in Figure 4. Thus side CD is labelled $a$ and DE as $b$, and they have lengths $a$, $b$. Inevitably the notation will become messy. Write

$$x_B - x_A = d_x, \quad y_B - y_A = d_y, \quad x_C - x_B = e_x, \quad y_C - y_B = e_y, \quad ..... \quad x_A - x_E = c_x, \quad y_A - y_E = c_y.$$

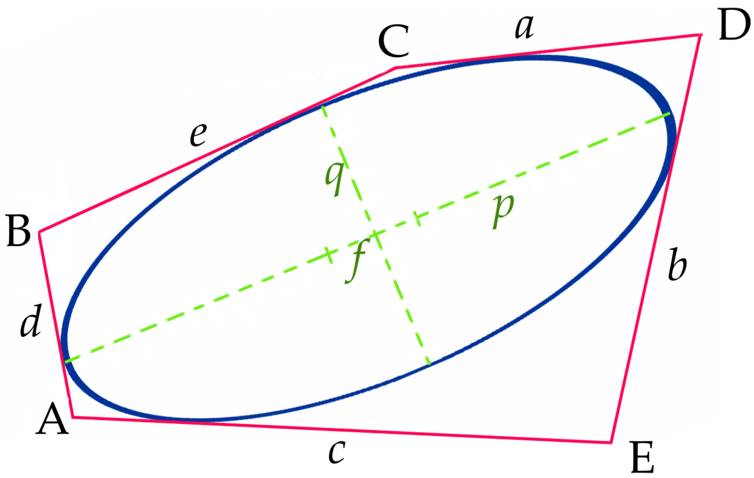

Figure 4: An ellipse inscribed in a general convex pentagon.

Let the gradients of the sides be AB: $m_d = d_y/d_x$, BC: $m_e = e_y/e_x$, CD: $m_a = a_y/a_x$, DE: $m_b = b_y/b_x$, and EA: $m_c = c_y/c_x$. The equations of the five sides are therefore

$$AB : y = m_d(x - x_A) + y_A, \qquad BC : y = m_e(x - x_B) + y_B, \qquad CD : y = m_a(x - x_C) + y_C,$$

$$DE : y = m_b(x - x_D) + y_D, \qquad EA : y = m_c(x - x_E) + y_E.$$

Substituting each of these expressions for $y$ in turn into Eq 1 gives five quadratic equations for $x$. We are interesting in the discriminants, which will be set to zero simultaneously. These are involved expressions, so I record only the one from line AB.

$$\Delta_d = -16 + 5p^2 + 5q^2 + 24x_0 - 9x_0^2 + 8y_0 - 6x_0y_0 - y_0^2 + 4p^2\cos 2\theta - 4q^2\cos 2\theta + 3p^2\sin 2\theta - 3q^2\sin 2\theta.$$

Numerical solution seems the only option. Here is a numerical example. The following pentagon has a shape similar to Figure 4:

$$A: \ (1, 1), \quad B: \ (0, 4), \quad C: \ (6, 7), \quad D: \ (12, 8), \quad E: \ (11, 0).$$

Using Mathematica I obtain the solution

$$p^2 = 34\cdot 8112, \qquad q^2 = 6\cdot 91477, \qquad x_0 = 6\cdot 17105, \qquad y_0 = 4\cdot 11184, \qquad \theta = 0\cdot 387969.$$

This is plotted in Figure 5. Note that we are offered only one solution of the five simultaneous equations. (There is a second solution, but it plots exactly the same because $p$ and $q$ are interchanged and $\theta$ adjusted accordingly.) So, unlike the triangle, there are no external figures being given which are tangent to the extended sides of the pentagon.

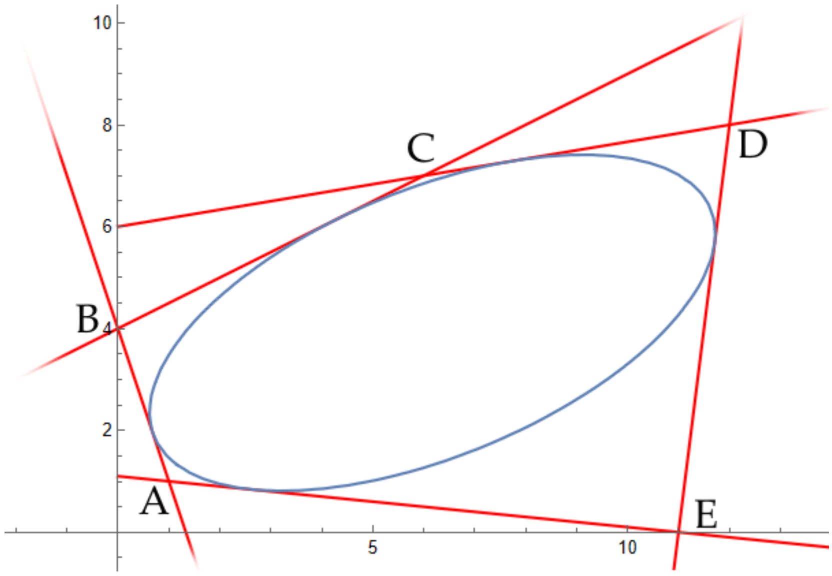

Figure 5: Ellipse calculated to fit the given pentagon.