The Spherical Triangle

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The Challenge : A sphere with unit radius centred at O: $(0, 0, 0)$ is cut by three non-parallel planes each of which passes through O. The spherical surface is thereby divided into eight shells, each a spherical triangle whose edges are great circles. Select one such triangle with vertices P, Q, R and obtain its properties in terms of the relative positions of P, Q, R and the angles between the cutting planes. In addition the planar triangle is drawn with vertices P, Q and R. Relate its properties to those of the spherical triangle which shares these three vertices.

Main features of the geometry

The four diagrams in Figure 1 show the two triangles in relation to the origin O of the sphere with unit radius and the three intersecting planes. We assume that the absolute position of the triangles around the sphere is not significant. Both triangles are completely specified by either the position vectors $\hat{\mathbf{p}}$, $\hat{\mathbf{q}}$, $\hat{\mathbf{r}}$ to the vertices, or by the angles θ, ξ, η between them as marked in the bottom left panel.

As a matter of notation, the mark ˆ over a vector will denote a unit vector. This will distinguish it from vectors which have not been normalised. Similarly angled brackets will mean that the vector function inside has been normalised to unit modulus. For instance, $<\hat{\mathbf{p}} - \hat{\mathbf{q}}> = (\hat{\mathbf{p}} - \hat{\mathbf{q}})/|\hat{\mathbf{p}} - \hat{\mathbf{q}}|$, with $|...|$ denoting the norm.

The unit vectors $\hat{\mathbf{d}}$ and $\hat{\mathbf{e}}$ lie in the tangent plane at P so the angle P between them is the dihedral[1] angle between the blue and red intersecting planes. This is also the angle between their normals. The normals pointing into the tetrahedron OPQR are given by the cross products $\hat{\mathbf{p}} \times \hat{\mathbf{r}}$ for the blue plane and $\hat{\mathbf{q}} \times \hat{\mathbf{p}}$ for the red. Recall that the magnitude of the cross (vector) product $|\hat{\mathbf{p}} \times \hat{\mathbf{r}}| = |\hat{\mathbf{p}}|.|\hat{\mathbf{r}}| \sin\theta = \sin\theta$. These vector relations hold:

$$\hat{\mathbf{d}}\cdot\hat{\mathbf{p}} = 0, \quad \hat{\mathbf{d}}\cdot(\hat{\mathbf{p}}\times\hat{\mathbf{r}}) = 0, \quad \hat{\mathbf{e}}\cdot\hat{\mathbf{q}} = 0, \quad \hat{\mathbf{e}}\cdot(\hat{\mathbf{p}}\times\hat{\mathbf{q}}) = 0, \quad \hat{\mathbf{d}}\cdot\hat{\mathbf{e}} = \cos P, \quad <\hat{\mathbf{p}}\times\hat{\mathbf{r}}>\cdot<\hat{\mathbf{p}}\times\hat{\mathbf{q}}> = \cos P.$$

$$\hat{\mathbf{f}}\cdot\hat{\mathbf{q}} = 0, \quad \hat{\mathbf{f}}\cdot(\hat{\mathbf{p}}\times\hat{\mathbf{q}}) = 0, \quad \hat{\mathbf{g}}\cdot\hat{\mathbf{q}} = 0, \quad \hat{\mathbf{g}}\cdot(\hat{\mathbf{q}}\times\hat{\mathbf{r}}) = 0, \quad \hat{\mathbf{f}}\cdot\hat{\mathbf{g}} = \cos Q, \quad <\hat{\mathbf{q}}\times\hat{\mathbf{r}}>\cdot<\hat{\mathbf{p}}\times\hat{\mathbf{q}}> = \cos Q.$$

$$\hat{\mathbf{k}}\cdot\hat{\mathbf{r}} = 0, \quad \hat{\mathbf{k}}\cdot(\hat{\mathbf{p}}\times\hat{\mathbf{r}}) = 0, \quad \hat{\mathbf{h}}\cdot\hat{\mathbf{r}} = 0, \quad \hat{\mathbf{h}}\cdot(\hat{\mathbf{q}}\times\hat{\mathbf{r}}) = 0, \quad \hat{\mathbf{k}}\cdot\hat{\mathbf{h}} = \cos R, \quad <\hat{\mathbf{p}}\times\hat{\mathbf{r}}>\cdot<\hat{\mathbf{q}}\times\hat{\mathbf{r}}> = \cos R.$$

$$-\hat{\mathbf{d}}\cdot\hat{\mathbf{k}} = \cos\theta, \quad -\hat{\mathbf{e}}\cdot\hat{\mathbf{f}} = \cos\xi, \quad -\hat{\mathbf{g}}\cdot\hat{\mathbf{h}} = \cos\eta. \tag{1}$$

[1] 'Dihedral' comes from the ancient Greek. 'Hedra' relates to sitting on a chair or throne, or figuratively as the seat of authority, and in this sense occurs in our word 'cathedral', the seat of the bishop. In a mathematical context hedra are the faces of a geometric solid, the surfaces on which it could rest.

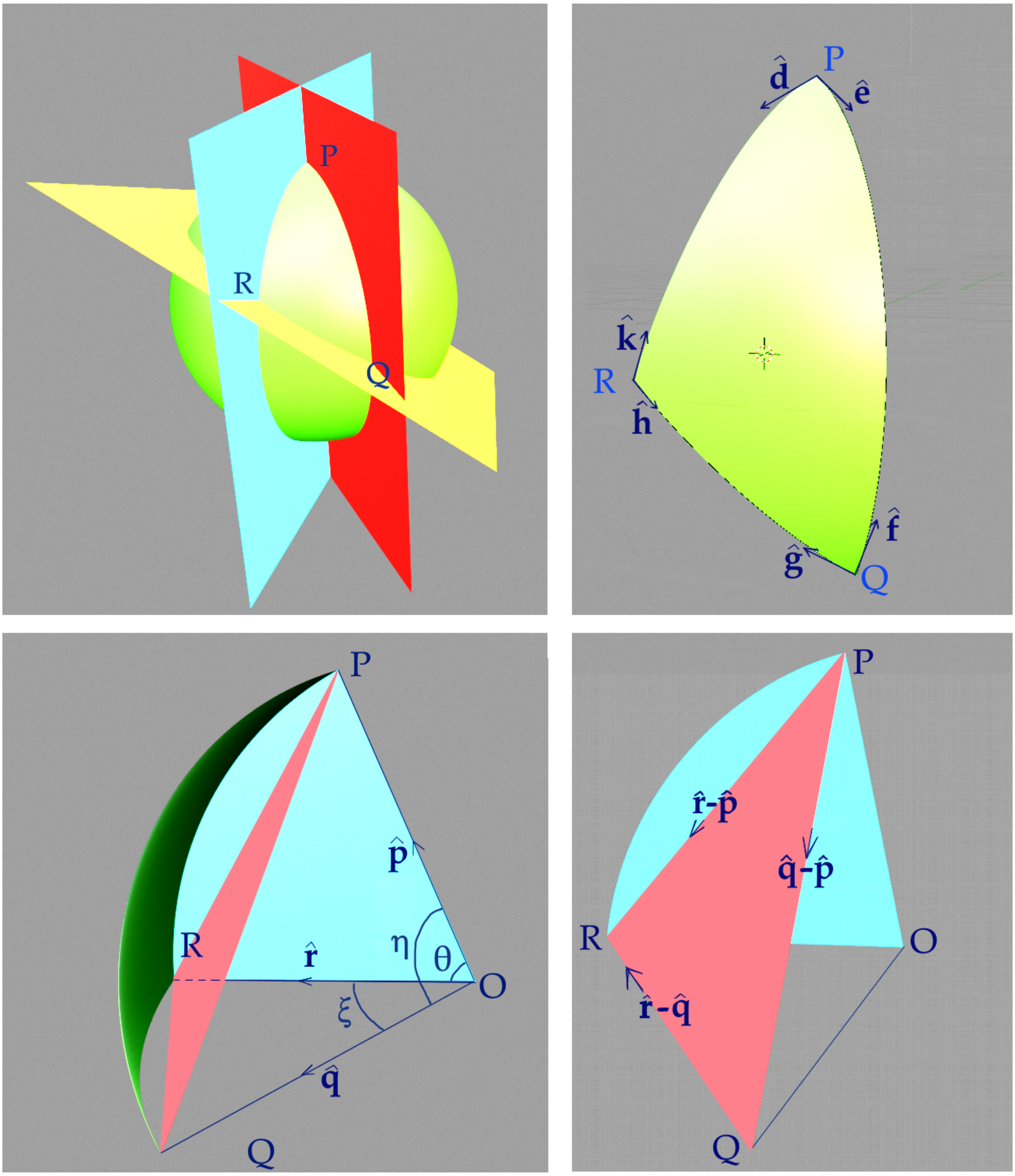

Figure 1: Four views of the spherical and planar triangles PQR cut from a unit radius sphere by three planes (blue, red, yellow). Unit position vectors $\hat{\mathbf{p}}$, $\hat{\mathbf{q}}$, $\hat{\mathbf{r}}$ are shown, as are the angles θ, ξ and η between them, and the three pairs of unit tangent vectors $\hat{\mathbf{d}}$ and $\hat{\mathbf{e}}$, $\hat{\mathbf{f}}$ and $\hat{\mathbf{g}}$, $\hat{\mathbf{h}}$ and $\hat{\mathbf{k}}$.

The planar triangle and tetrahedron

The planar triangle is easier to analyse than the spherical so we deal with it first. It is one face of the tetrahedron OPQR. Its three edges are the vectors $\hat{\mathbf{q}} - \hat{\mathbf{p}}$ from P to Q, $\hat{\mathbf{r}} - \hat{\mathbf{p}}$ from P to R, and $\hat{\mathbf{r}} - \hat{\mathbf{q}}$ between Q and R. The norm of each of these is the length of the side.

Denote the three angles by P', Q', R' to distinguish them from the angles in the spherical triangle. Using normalised vectors along the sides, these angles are given by

$$< \hat{\mathbf{q}} - \hat{\mathbf{p}} > \cdot < \hat{\mathbf{r}} - \hat{\mathbf{p}} > = \cos P', \quad < \hat{\mathbf{p}} - \hat{\mathbf{q}} > \cdot < \hat{\mathbf{r}} - \hat{\mathbf{q}} > = \cos Q', \quad < \hat{\mathbf{r}} - \hat{\mathbf{p}} > \cdot < \hat{\mathbf{r}} - \hat{\mathbf{q}} > = \cos R'. \quad (2)$$

The area $\mathcal{A}'$ of PQR is 'half the base × the perpendicular height'. Taking QR as the base, the projection of side PR onto the base has length $(\hat{\mathbf{r}} - \hat{\mathbf{p}}) \cdot < \hat{\mathbf{r}} - \hat{\mathbf{q}} >$. Calling this g and h the height from the base to vertex P, Pythagoras' theorem gives $h^2 = |\hat{\mathbf{r}} - \hat{\mathbf{p}}|^2 - g^2$. Then

$$\mathcal{A}' = \tfrac{1}{2}|\hat{\mathbf{r}} - \hat{\mathbf{q}}|\sqrt{|\hat{\mathbf{r}} - \hat{\mathbf{p}}|^2 - \frac{[(\hat{\mathbf{r}} - \hat{\mathbf{p}}) \cdot \hat{\mathbf{r}} - \hat{\mathbf{q}})|^2}{|\hat{\mathbf{r}} - \hat{\mathbf{q}}|^2}} = \tfrac{1}{2}\sqrt{|\hat{\mathbf{r}} - \hat{\mathbf{p}}|^2 \, |\hat{\mathbf{r}} - \hat{\mathbf{q}}|^2 - |(\hat{\mathbf{r}} - \hat{\mathbf{p}}) \cdot (\hat{\mathbf{r}} - \hat{\mathbf{q}})|^2}. \quad (3)$$

An alternative formula is the modulus of the cross produce of any two sides. Thus

$$\mathcal{A}' = \tfrac{1}{2}|(\hat{\mathbf{r}} - \hat{\mathbf{p}}) \times (\hat{\mathbf{q}} - \hat{\mathbf{p}})| = \tfrac{1}{2}|(\hat{\mathbf{r}} - \hat{\mathbf{q}}) \times (\hat{\mathbf{p}} - \hat{\mathbf{q}})| = \tfrac{1}{2}|(\hat{\mathbf{p}} - \hat{\mathbf{r}}) \times (\hat{\mathbf{q}} - \hat{\mathbf{r}})|. \quad (4)$$

These arise because in general $|\mathbf{s} \times \mathbf{t}| = |\mathbf{s}|\,|\mathbf{t}|.\sin(\text{angle between})$, so if $|\hat{\mathbf{r}} - \hat{\mathbf{q}}|$ is the base, $|\hat{\mathbf{p}} - \hat{\mathbf{q}}| \sin Q'$ is the perpendicular height.

These expressions may look compact in vector notation, but become long and cumbersome when written out as co-ordinates within an xyz co-ordinate frame. When it comes to making a specific calculation, it is probably tidiest to calculate the vector contributions along the way rather than have a final grand algebraic formula into which substitutions are made. To illustrate this discussion, therefore, I propose to carry through calculations with a specific but otherwise arbitrary triangle. Its defining properties are given in Table 1. Since the sphere has unit radius, all the position vectors are unit vectors. Several scalar and vectors products are listed in Table 2. Many of these confirm the relations at Eq 1. Norm refers to the length of the vector. Angles and side lengths are marked in Figure 3. From Eq 2 the angles of triangle PQR are $P' = 32 \cdot 8°$, $Q' = 49 \cdot 2°$, $R' = 98 \cdot 1°$ and of course they sum to 180°. There are no such universal relations between the planar or solid angles of the tetrahedron. The familiar Sine and Cosine formulae are readily verified. Write length QR $= p$, PR $= q$, PQ $= r$. Then

$$\frac{\sin P}{p} = \frac{\sin Q}{q} = \frac{\sin R}{r} = 0 \cdot 6838, \qquad \cos R = \frac{p^2 + q^2 - r^2}{2pq}.$$

In the tetrahedron OPQR the angles between edges at O are

$$\hat{\mathbf{p}} \cdot \hat{\mathbf{r}} = \cos\theta, \qquad \hat{\mathbf{p}} \cdot \hat{\mathbf{q}} = \cos\xi, \qquad \hat{\mathbf{q}} \cdot \hat{\mathbf{r}} = \cos\eta. \quad (5a)$$

Since the sphere has unit radius, these angles in radians are numerically equal to the arc lengths of the edges of the spherical triangle. In the example the angles between the edges emanating from O are

$$\mathbf{p} \cdot \mathbf{r} = 0 \cdot 3873 \implies \theta = 67 \cdot 2°, \quad \mathbf{p} \cdot \mathbf{q} = -0 \cdot 0491 \implies \xi = 92 \cdot 8°, \quad \mathbf{q} \cdot \mathbf{r} = 0 \cdot 6860 \implies \eta = 46 \cdot 7°. \quad (5b)$$

	x	y	z
O	0	0	0
P	$0\cdot2647$	$-0\cdot3597$	$0\cdot8945$
Q	$0\cdot4734$	$-0\cdot7321$	$-0\cdot4894$
R	$-0\cdot0466$	$-0\cdot9977$	$0\cdot0455$

Table 1: Co-ordinates of the three vertices of the example planar and spherical triangles, being also the components of the position vectors $\hat{\mathbf{p}}$, $\hat{\mathbf{q}}$, $\hat{\mathbf{r}}$.

	x	y	z	Norm		x	y	z
$\hat{\mathbf{r}}-\hat{\mathbf{p}}$	$-0\cdot3113$	$-0\cdot6379$	$-0\cdot8490$	$1\cdot107$	$<\hat{\mathbf{r}}-\hat{\mathbf{p}}>$	$-0\cdot2813$	$-0\cdot5764$	$-0\cdot7672$
$\hat{\mathbf{q}}-\hat{\mathbf{p}}$	$0\cdot2087$	$-0\cdot3724$	$-1\cdot3839$	$1\cdot448$	$<\hat{\mathbf{q}}-\hat{\mathbf{p}}>$	$0\cdot1441$	$-0\cdot2571$	$-0\cdot9556$
$\hat{\mathbf{r}}-\hat{\mathbf{q}}$	$-0\cdot5200$	$-0\cdot2656$	$0\cdot5349$	$0\cdot792$	$<\hat{\mathbf{r}}-\hat{\mathbf{q}}>$	$-0\cdot6567$	$-0\cdot3354$	$0\cdot6755$
$\hat{\mathbf{p}}\times\hat{\mathbf{r}}$	$0\cdot8761$	$-0\cdot0538$	$-0\cdot2808$	$0\cdot922$	$<\hat{\mathbf{p}}\times\hat{\mathbf{r}}>$	$0\cdot9506$	$-0\cdot0583$	$-0\cdot3048$
$\hat{\mathbf{q}}\times\hat{\mathbf{p}}$	$-0\cdot8309$	$-0\cdot5530$	$0\cdot0235$	$0\cdot998$	$<\hat{\mathbf{q}}\times\hat{\mathbf{p}}>$	$-0\cdot8323$	$-0\cdot5539$	$0\cdot0235$
$\hat{\mathbf{r}}\times\hat{\mathbf{q}}$	$0\cdot5216$	$-0\cdot0012$	$0\cdot5064$	$0\cdot727$	$<\hat{\mathbf{r}}\times\hat{\mathbf{q}}>$	$0\cdot7174$	$-0\cdot0017$	$0\cdot6966$

Table 2: Components of vectors derived from $\hat{\mathbf{p}}$, $\hat{\mathbf{q}}$, $\hat{\mathbf{r}}$. $<..>$ denotes a normalised vector. The cross product vectors point inwards into the tetrahedron.

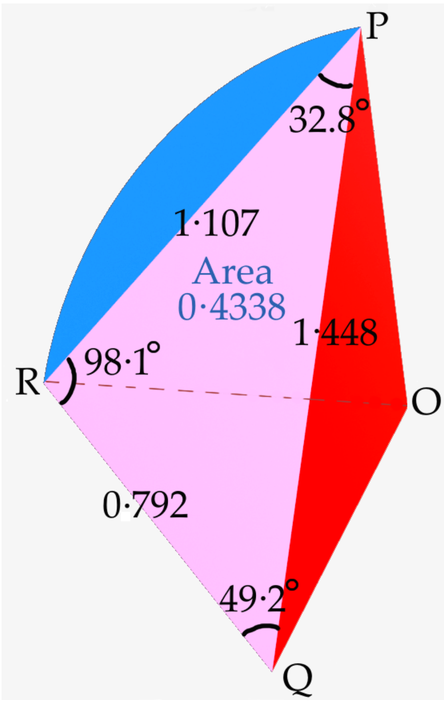

Figure 2: Angles, side lengths and area of planar triangle PQR.

These are three of the nine internal angles between edges of the tetrahedron. In radians these are the lengths of the arc sides of the spherical triangle.

The outwards normal to the planar triangle is given by

$$(\hat{\mathbf{r}}-\hat{\mathbf{p}})\times(\hat{\mathbf{q}}-\hat{\mathbf{p}}) \;=\; (0\cdot5667,\ -0\cdot6080,\ 0\cdot2491), \quad \text{with modulus } 0\cdot8677.$$

The area of the planar triangle PQR is half this, $0\cdot4338$ square units. The formula Eq 3 gives the same value. The unit normal $\hat{\mathbf{n}}$ is $(0\cdot6531, -0\cdot7007, 0\cdot2870)$, so the other dihedral angles in the tetrahedron, those involving triangle PQR, are

$$\text{with face OPR:}\quad \hat{\mathbf{n}}\times <\hat{\mathbf{p}}\times\hat{\mathbf{r}}> \Longrightarrow 55\cdot0°,$$

$$\text{with face OPQ:}\quad \hat{\mathbf{n}}\times <\hat{\mathbf{q}}\times\hat{\mathbf{p}}> \Longrightarrow 98\cdot6°,$$

$$\text{with face OQR:}\quad \hat{\mathbf{n}}\times <\hat{\mathbf{q}}\times\hat{\mathbf{r}}> \Longrightarrow 48\cdot0°,$$

For completeness the areas of the other three faces of the tetrahedron with vertices O, P, Q, R are

$$\text{PQO} : 0\cdot4992, \quad \text{PRO} : 0\cdot4608, \quad \text{ORQ} : 0\cdot3635.$$

Another parameter of the triangle is its centroid, G, which is at the mean position over the surface, its centre of gravity, and at the intersection of the medians which joint each vertex to the mid-point of its opposite side. We have

$$\mathbf{g} = \frac{1}{3}(\hat{\mathbf{p}}+\hat{\mathbf{q}}+\hat{\mathbf{r}}) = (0\cdot2305,\ -0\cdot6965,\ 0\cdot1502).$$

We can obtain some further properties of this tetrahedron. Its volume is given by the vector 'box product'. For any three non-coplanar vectors, $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, $\mathbf{u}\times\mathbf{v}$ is the numerically equal to the area of the parallelogram with sides $\mathbf{u}$ and $\mathbf{v}$ and also a vector normal to it. Taking the scalar produce of this normal with the third vector gives the perpendicular height of the parallelepiped. Since the tetrahedron has a triangular base which is one half the base of the parallelogram, and since the volume of a solid cone is a third that of the prism with the same base, the volume of the tetrahedron is

$$V' = \frac{1}{6}(\hat{\mathbf{p}}\times\hat{\mathbf{q}}\cdot\hat{\mathbf{r}}). \tag{6}$$

The volume of the parallelepiped is $0\cdot5915$ cubic units so that of the tetrahedron is $0\cdot0986$.

Spherical geometry

Before examining the spherical triangle, let us reflect on some of the differences of spherical geometry from planar Euclidean geometry. Take the concept of a triangle. All lines on the surface of a sphere are by their nature curved, but the geodesics – the shortest paths between two points – are the great circles. To every great circle are a myriad of small circles corresponding to the circles of latitude parallel to the great circle as equator. There is a unique great circle C passing through any two chosen points on the sphere, but it is possible to join the points with any number of small circles, each a latitude of a great circle with the same poles as C. Similarly many triangles could be drawn through three points, but only one has three sides which are great circles, as with PQR in this article. Some writers refer to these as 'strict' spherical triangles; we may also call them 'authentic' or 'great', and the others as 'quasi-triangles' or 'lesser triangles'.

Another difference from planar geometry is the awkwardness of visualising surface area. On a planar surface area will be measured in square units of length, such as square metres, and

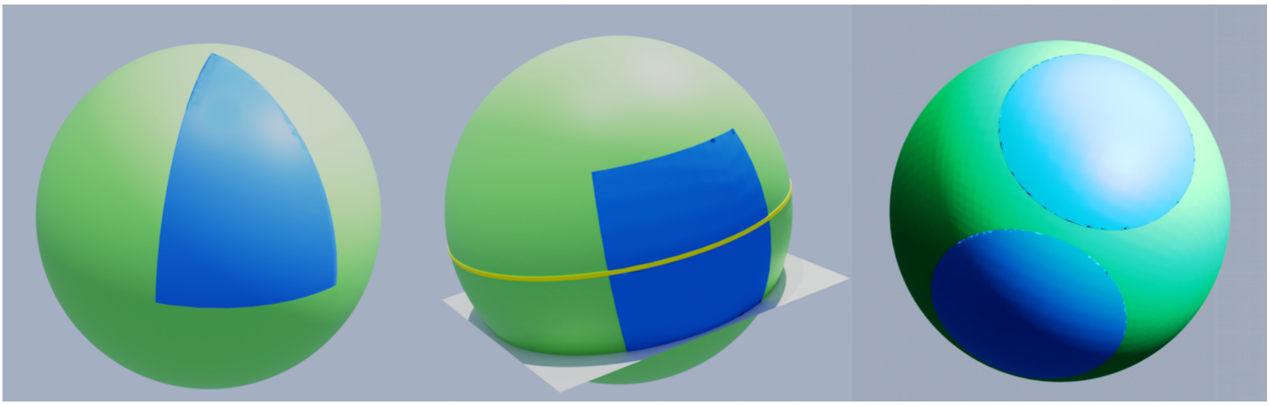

Figure 3: Three depictions of unit area on a sphere with radius 1. The solid angle subtended by each blue shape is 1 steradian. Central panel shows equator and plane at 30° latitude.

we can readily draw a reference square with sides one unit long and four internal right angles. If this is folded and cut in half, the two halves can be placed into a half-by-two rectangle, and we readily see that this has the same unit area. On a sphere a unit of surface area is less easily visualised and compared with other areas. The surface area of a sphere is $4\pi R^2$ square units. If the radius is 1, this is about $12 \cdot 5$ square units. This value is found by integration of a surface element in spherical polar co-ordinates θ, ϕ as noted in Appendix 1. One square unit of area is naturally shown as a segment of a hemisphere – the smaller space bounded by two great circles of longitude 1 radian apart in azimuthal angle ϕ, and their equator. This shape is the spherical triangle shown in the left panel of Figure 3. The solid angle subtended at the centre of the sphere by any patch of unit area is one steradian – a solid radian. If the radius is 1, surface area and solid angle are numerically the same.

Figure 3 shows two other blue surface patches whose areas are also 1 square unit. The central panel is a quasi-rectangle which great circles along two opposite sides and lines of latitude along the other. In Appendix 1, Eq A1.2, the area of a spherical cap on the unit sphere is shown to be $2\pi(1 - \cos\theta_0)$ where θ_0 is the angle from the pole (the centre of the cap) to the rim. The area of a sector 1 radian in ϕ is therefore $1 - \cos\theta_0$ so the small circle at $\theta = 60°$ divides the triangle of the left panel of Figure 3 into two parts of equal area. In the central panel the rectangular shape has been mirrored in the equatorial plane to make a combined area of 1 square unit. The right-hand panel shows two caps each formed by a cone of semi-angle $32 \cdot 8°$ with centre at O intersecting the sphere. $\theta = 32 \cdot 8°$ solves the equation $2\pi(1-\cos\theta_0) = 1$. So the full angle of the cone subtending a unit-area small circle is $65 \cdot 6°$, to be compared with 1 radian = $57 \cdot 3°$ on a planar circle.

These are probably the three simplest unit-area shapes. We turn now to the triangle PQR of Figure 1.

Area of a spherical triangle

Below I give my calculation of the area of a spherical triangle by integration. However, there is a clever, elegant way of deriving the essential formula which I first reproduce here.

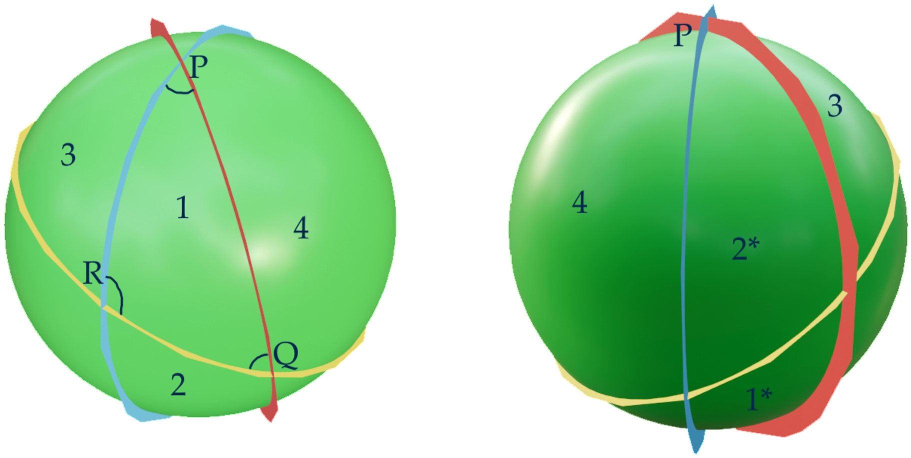

Figure 4: The unit sphere cut by three central planes to form triangle PQR which lies in three lunes. Left: front view. Right: back view.

Figure 4 is similar to the first panel in Figure 1, with the three cutting planes shrunk almost to great circles. Each pair of planes forms a 'lune' – a sector of the surface between two great circles. The area of the whole unit sphere is 4π, and the area of a lune between great circles ϕ radians apart is 2ϕ square units. Our example triangle PQR forms a part of three lunes: between the blue and red planes, the red and yellow, and the yellow and blue. Let the area of numbered region n be An. PQR is region 1 so we want to calculate A1. Then

$$A1 + A2 = 2P, \qquad A1 + A3 = 2Q. \qquad A1 + A4 = 2R.$$

The regions marked 1* and 2* seen at the back of the sphere are equivalent triangles in the rear hemisphere, and are congruent to 1 and 2. It can be seen that regions 1, 3, 4 and 2* together exactly cover a hemisphere. Therefore, adding the angles

$$2P + 2Q + 2R \;=\; 3A1 + A2 + A3 + A4 \;=\; 2A1 + 2\pi \quad \text{so} \quad A1 \;=\; P + Q + R - \pi. \tag{7}$$

The area is said to be equal to the 'spherical angular excess', over the 180° of a planar triangle. We met a simple example of this in the left panel of Figure 3 where the angles at the equator are both 90°. It will be clear that the area and hence the side lengths cannot be changed without changing the angles. Therefore a triangle with three given angles on a unit sphere is unique. Similar, non-congruent triangles do not exist in spherical geometry.

Sides and angles

Just as the planar triangle is specified by the length of its sides and the angles P', Q', R' between them, so the spherical triangle is characterised by the arc lengths of its edges and the angles P, Q, R between them. These are the angles between the cutting planes and are found from expressions such as

$$< \hat{\mathbf{p}} \times \hat{\mathbf{r}} > \cdot < \hat{\mathbf{p}} \times \hat{\mathbf{q}} > = 0 \cdot 7660 \quad \Longrightarrow \quad P = 40.0°, \quad 0 \cdot 6981 \text{ radians.} \tag{8a}$$

The other two dihedral angles are $Q = 54\cdot 6° = 0\cdot 95235$ radians, $R = 118\cdot 0° = 2\cdot 0599$ radians. The components of these cross products are equal to the coefficients of the cutting planes, so from Table 2

$$\text{plane OPR}: \quad 0\cdot 9506x - 0\cdot 0583y - 0\cdot 3048z = 0,$$

$$\text{plane OPQ}: \quad -0\cdot 8323x - 0\cdot 5539y + 0\cdot 0235z = 0,$$

$$\text{plane OQR}: \quad 0\cdot 7174x - 0\cdot 0017y + 0\cdot 6966z = 0.$$

Referring back to Figure 1, the unit tangent vectors directed along the great circle edges, $\hat{\mathbf{d}}$ to $\hat{\mathbf{k}}$, also characterise the triangle. The three components of each of these can be calculated in at least two ways. First, from the relations in Eq 1 together with them being unit vectors. Three independent equations for each vector are required, so for $\hat{\mathbf{d}}$

$$\hat{\mathbf{d}}\cdot\hat{\mathbf{p}} = 0, \qquad \hat{\mathbf{d}}\cdot(\hat{\mathbf{p}}\times\hat{\mathbf{r}}) = 0, \qquad |\hat{\mathbf{d}}| = 1.$$

The first two give simultaneous linear equations in two components. The third equation offers a choice of sign and both need to be considered. In general d_y and d_z can be expressed in terms of d_x, and d_x chosen to normalise to a unit vector:

$$d_y = dx\left(\frac{p_y(p_x r_x + p_z r_z) - (p_x^2 + p_z^2)r_y}{p_x(p_y r_y - p_z r_z) - (p_y^2 + p_z^2)r_x}\right), \qquad dz = \frac{-p_x d_x - p_y d_y}{p_z}.$$

The second method is to use the triple vector product relations

$$\hat{\mathbf{d}} = \frac{(\hat{\mathbf{p}}\times\hat{\mathbf{r}})\times\hat{\mathbf{p}})}{|...|}, \qquad \hat{\mathbf{e}} = \frac{(\hat{\mathbf{p}}\times\hat{\mathbf{r}})\times\hat{\mathbf{p}})}{|...|},$$

where $|...|$ denotes the modulus of the numerator.

For our example the numbers are

$$\begin{aligned} \hat{\mathbf{d}} &= (-0\cdot 1618, -0\cdot 9312, -0\cdot 3266), & \hat{\mathbf{e}} &= (\ 0\cdot 4871, -0\cdot 7508, -0\cdot 4461), \\ \hat{\mathbf{f}} &= (\ 0\cdot 2883, -0\cdot 3962,\ 0\cdot 8717), & \hat{\mathbf{g}} &= (-0\cdot 5109, -0\cdot 6810,\ 0\cdot 5245), \\ \hat{\mathbf{h}} &= (\ 0\cdot 6951, -0\cdot 0652, -0\cdot 7160), & \hat{\mathbf{k}} &= (\ 0\cdot 3068,\ 0\cdot 0291,\ 0\cdot 9513.) \end{aligned}$$

The scalar product of $\hat{\mathbf{d}}$ and $\hat{\mathbf{e}}$ gives the angle P of the spherical triangle, and this equals the angle between the blue and red planes. The three angles of the triangle found from $\hat{\mathbf{d}}\cdot\hat{\mathbf{e}}$, $\hat{\mathbf{f}}\cdot\hat{\mathbf{g}}$ and $\hat{\mathbf{h}}\cdot\hat{\mathbf{k}}$ are

$$P = 40\cdot 0° = 0\cdot 698 \text{ rads}, \qquad Q = 54\cdot 6° = 0\cdot 952 \text{ rads}, \qquad R = 118\cdot 0° = 2\cdot 060 \text{ rads} \quad (8b)$$

and of course agree with the values found from the normals at Eq 8a. Their sum is $212\cdot 6° = 3\cdot 711$ radians, an excess of $32\cdot 6° = 0\cdot 569$ radians over the planar triangle PQR, so by Eq 7 this is the area of the spherical triangle.

The side arc lengths are numerically equal to the angles θ, ξ, η in radians. These were found for the tetrahedron at Eq 5b, and here are obtained again from $-\hat{\mathbf{d}}\cdot\hat{\mathbf{k}} = +\hat{\mathbf{p}}\cdot\hat{\mathbf{r}}$, $-\hat{\mathbf{e}}\cdot\hat{\mathbf{f}} = +\hat{\mathbf{p}}\cdot\hat{\mathbf{q}}$ and $-\hat{\mathbf{g}}\cdot\hat{\mathbf{h}} = +\hat{\mathbf{q}}\cdot\hat{\mathbf{r}}$. They are

$$\text{arc PR} = 1\cdot 173 \equiv \theta\ (67\cdot 2°), \quad \text{PQ} = 1\cdot 620 \equiv \xi\ (92\cdot 8°), \quad \text{QR} = 0\cdot 815 \equiv \eta\ (46\cdot 7°). \quad (9)$$

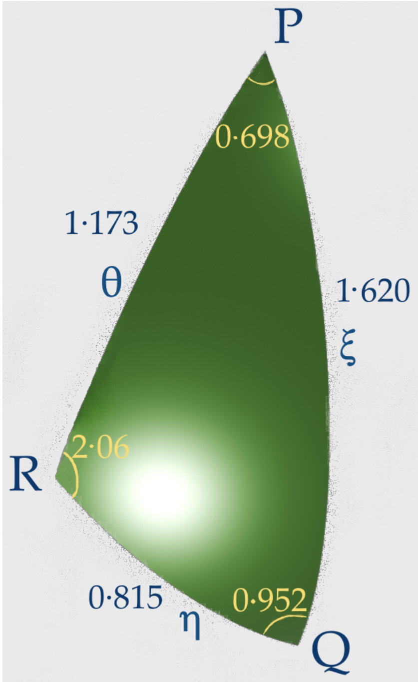

Figure 5: Dimensions of the example spherical triangle. Angles in radians.

So here is a complete description of the spherical triangle. The dimensions are written in Figure 5, which can be compared with Figure 2 for the planar triangle.

There are only 9 co-ordinates to vertices P, Q, R so, as with the planar triangle, there must be relations between the angles and side lengths similar to the familiar Sine and Cosine formulae. Bearing in mind that the edge arc lengths are numerically equal to the angles θ, ξ, η at O in the tetrahedron, we can examine $\sin P/\eta$, $\sin Q/\theta$ and $\sin R/\xi$ as in Figure 1. These are not equal because the denominators increase too rapidly with angle size. The sine function compensates for this and we find numerically that

$$\frac{\sin P}{\sin \eta} = \frac{\sin Q}{\sin \theta} = \frac{\sin R}{\sin \xi} = 0 \cdot 8838.$$

This corresponds to a vector identity. An algebraic proof is given in Appendix 2 where it is shown that

$$\frac{\sin P}{\sin \eta} = \frac{\sin Q}{\sin \theta} = \frac{\sin R}{\sin \xi} = \frac{\hat{\mathbf{p}} \cdot (\hat{\mathbf{q}} \times \hat{\mathbf{r}})}{|\hat{\mathbf{p}} \times \hat{\mathbf{q}}||\hat{\mathbf{q}} \times \hat{\mathbf{r}}||\hat{\mathbf{r}} \times \hat{\mathbf{p}}|}, \qquad (10a)$$

a constant property of the triangle. If we allow for the moment the spherical triangle to be have vertices A, B, C, angles A, B, C and opposite side arc lengths a, b, c, this becomes more familiar as

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}. \qquad (10b)$$

Two other 'rules' we can look for are the equivalents of the Cosine rule and Pythagoras's theorem. The Cosine rule relates the lengths of two sides to the angle between them. For a planar triangle ABC it is readily proved by taking the sides adjacent to vertex C to be **a** and **b**, in which case $\mathbf{c} = \mathbf{a} - \mathbf{b}$. Then

$$\mathbf{c}^2 = \mathbf{c}\cdot\mathbf{c} = \mathbf{a}^2 - 2\mathbf{a}\cdot\mathbf{b} + \mathbf{b}^2 = \mathbf{a}^2 + \mathbf{b}^2 - 2|\mathbf{a}||\mathbf{b}|\cos C.$$

We can derive a type of Cosine rule by using this relation on the triangle PST which is tangent to the sphere at P, and on the other three sides of the tetrahedron it forms with the cutting planes through O, as shown in Figure 6. We use the facts that i) the arc lengths are equal to the angles θ, ξ and η, and ii) that triangles OPS and OPT each have a right angle at P. With the notation in Figure 6

$$k^2 = s^2 + t^2 - 2st\cos P = m^2 + n^2 - 2mn\cos\eta,$$

$$\cos\theta = \frac{1}{m}, \quad \cos\xi = \frac{1}{n}, \quad \sin\theta = \frac{s}{m}, \quad \sin\xi = \frac{t}{n}, \quad m^2 = s^2 + 1, \quad n^2 = t^2 + 1.$$

$$-st\cos P = 1 - mn\cos\eta = -mn\sin\theta\sin\xi\cos P,$$

$$-\sin\theta\sin\xi\cos P = \frac{1}{mn} - \cos\eta,$$

$$\text{Cosine Rule}: \quad \cos\eta = \cos\theta\,\cos\xi + \sin\theta\,\sin\xi\,\cos P. \tag{11a}$$

Appendix 2 gives a proof of this using vector identities. In more familiar notation

$$\text{Cosine Rule}: \quad \cos c = \cos a\,\cos b + \sin a\,\sin b\,\cos C. \tag{11b}$$

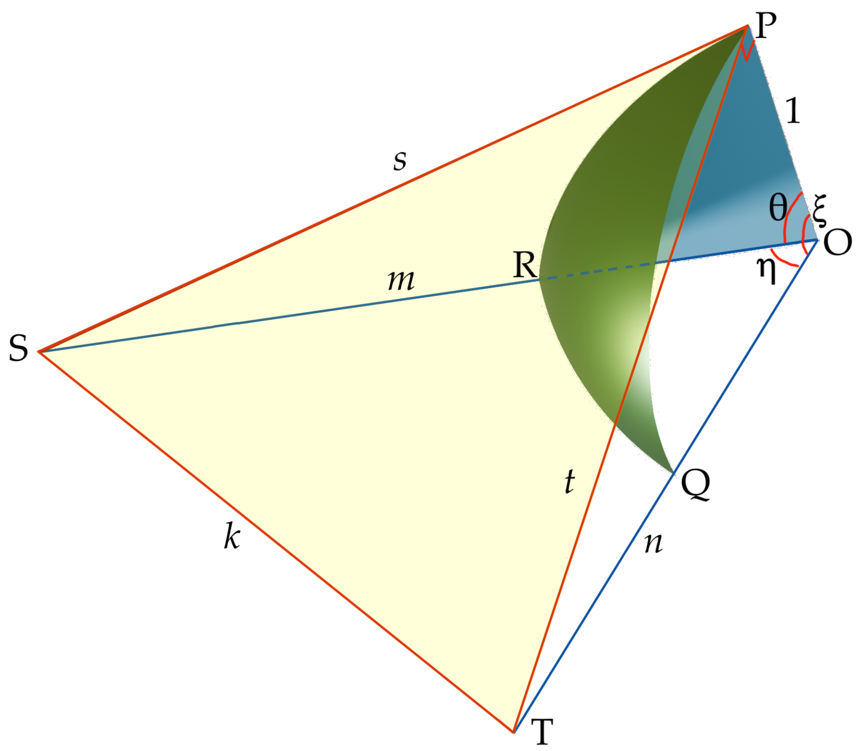

Figure 6: A planar triangle at P spanned by tangent vectors $\hat{\mathbf{d}}$ and $\hat{\mathbf{e}}$, and the tetrahedron OPST.

Pythagoras's theorem relates the three side lengths in a right-angled triangle to each other. It is recovered from the Cosine rule for planar triangles ABC by setting $C = \pi/2$ and we do the same with the Cosine rule for spherical triangles to obtain for $P = 90°$

$$\cos c = \cos a \cos b. \tag{12}$$

Spherical triangles have a second form of the Cosine rule not shared with planar ones. The 18th century pioneers of spherical trigonometry devised a second triangle from each given spherical one. They probably were intrigued by the fact that the arc length of a side is numerically equal to the angle it subtends at O, so lengths and angles seem almost interchangeable. They constructed a 'dual' triangle in which the values of the angles and of the sides were swapped. To be precise, they replaced angle A by arc length $\pi - a$ and side length b by angle $\pi - B$. This applies only to triangles in which no side is longer than π. Using $\sin(\pi - a) = \sin a$ and $\cos(\pi - a) = -\cos a$, the dual of Eq 11b is

$$\text{Dual Cosine Rule :} \qquad \cos C = -\cos A \cos B + \sin A \sin A \cos c. \tag{13}$$

This type of relation occurs of other non-Euclidean geometries. For example, in hyperbolic geometry the Cosine rules are

$$\cosh c = \cosh a \cosh b - \sinh a \sinh b \cos C, \qquad \cos C = -\cos A \cos B + \sin A \sin B \cosh C.$$

Rotation to standard spherical co-ordinates

In the next section I determine the area of the spherical triangle PQR by integration using spherical coordinates. The familiar spherical coordinate system r, θ, ϕ is not naturally configured to cope with integration over a region bounded by other circles, great or small, which do not share the same two poles. To calculate the area, therefore, one approach is first to rotate it to a standard configuration matched to the spherical co-ordinate frame.

The standard spherical co-ordinate frame as used by physicists has the z axis vertical and defines the North pole. The polar angle (the 'co-latitude') θ is measured from the z axis towards the $x - y$ plane. In this plane the azimuth (longitude) ϕ is measured from x towards y. An orthonormal transformation will rotate any vector about the origin O as centre with no stretching, and if its determinant is +1, there will be no accompanying reflection which would flip a shape into its mirror image. In a companion article on perspective, also on *www.mathstudio,.co.uk*, I show in some detail how a 3-by-3 matrix can be constructed with these properties. We wish to rotate vertex P of triangles PRQ to A: $(0, 0, 1)$, R to C: $(c_x,\ 0,\ c_z)$ and Q to B: $(b_x,\ b_y,\ b_z)$. The effect of this is shown in Figure 7. Here the unit sphere has been shrunk slightly to allow the vertices of the original and rotated planar triangles to stick out. The right panel has greater shrinkage. The position vectors of A, B, C are $\hat{\mathbf{a}}$, $\hat{\mathbf{b}}$, $\hat{\mathbf{c}}$.

The matrix effecting this rotation for our example is

$$\begin{pmatrix} -0\cdot 1618 & -0\cdot 9312 & -0\cdot 3266 \\ 0\cdot 9505 & -0\cdot 0583 & -0\cdot 3047 \\ 0\cdot 2648 & -0\cdot 3599 & 0\cdot 8948 \end{pmatrix}. \tag{14}$$

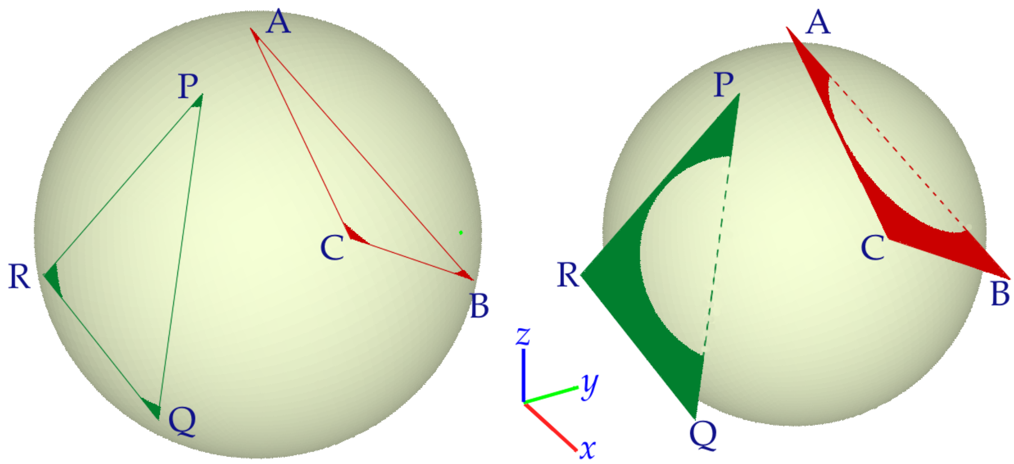

Figure 7: The original planar triangle PQR and its rotated version ABC. The unit sphere has been shrunk to show the vertices.

This has been derived by solving a number of simultaneous equations which express the facts that

- the dot (scalar) product of any two rows of the matrix is zero,
- the dot product of any two columns of the matrix is zero,
- the sum of squares of the elements in any row or column is 1,
- the determinant of the matrix is +1.

I obtained the matrix above by a sequence of single steps, but also have written a computer program to solve this system. Pseudo-code is given in Appendix 3. Where there is the option of a + or − sign, both have to be considered and checked against the determinant criterion. The rotated positions of P, Q, R are

	x	y	z
O	0	0	0
P → A	0	0	1
Q → B	0 · 7649	0 · 6417	−0 · 0491
R → C	0 · 9217	0	0 · 3874

Table 3: Co-ordinates of the three vertices P, Q, R of the example planar and spherical triangles after rotation to A, B, C.

The reader may have noticed that the top row in the matrix, Eq 14, is numerically the components of vector $\hat{\mathbf{d}}$. The reason is simply that $\hat{\mathbf{d}}$ rotates to $(1, 0, 0)$. $\hat{\mathbf{d}} \cdot \hat{\mathbf{d}} = 1$ and the other two rows are orthogonal to $\hat{\mathbf{d}}$. Similarly $\hat{\mathbf{e}}$ rotates to $(0 \cdot 7660,\ 0 \cdot 6427,\ 0)$ at right-angles to the z axis at $\phi = 40°$.

Equipped with a mechanism for rotation, we rotate triangle PQR to standard position using the rotation matrix above, and relabel it ABC in preparation for integrating its area.

Area by integral calculus

To determine the area $\mathcal{A}$ it is necessary to determine in spherical co-ordinates the equations of the great circles which form its edges. AC is clearly $\phi = 0$ with θ from 0 to $\arctan(c_x/c_z)$. For our example this $67\cdot 2°$, as can also be obtained from the scalar product $\hat{\mathbf{a}}\cdot\hat{\mathbf{c}}$. We noted in a previous section that, following Eq 1, the angles between the intersecting planes are the angles between their normals, obtained from expressions such as

$$<\hat{\mathbf{a}}\times\hat{\mathbf{c}}>\cdot<\hat{\mathbf{a}}\times\hat{\mathbf{b}}> = 0\cdot 7551 \implies A = 40\cdot 0°.$$

The other two dihedral angles are $B = 54\cdot 6°$ and $C = 118\cdot 0°$. These are also the interior angles of the spherical triangle and they add to $212\cdot 6° = 3\cdot 709$ radians, an excess of $0\cdot 572$ radians over the angle sum π for a planar triangle. The excess area is obviously due to the curved sides bowing outwards.

The great circle containing BC is less straightforward. It is the intersection of the plane OBC with the sphere $x^2+y^2+z^2 = 1$. Notate the normalised cross product $<\hat{\mathbf{c}}\times\hat{\mathbf{b}}>$ as $(v_x,\, v_y,\, v_z)$ so $v_x^2+v_y^2+v_z^2 = 1$. This is the normal to the plane pointing into the hemisphere containing P. Its components are also the coefficients of the plane $v_x x + v_y y + v_z z = 0$, from which $z = -(v_x x + v_y y)/v_z$. In the example this is $(-0\cdot 3420,\, 0\cdot 4700,\, 0\cdot 8137)$ and

$$\text{plane OBC}: \quad -0\cdot 3420\,x + 0\cdot 4700\,y + 0\cdot 8137\,z = 0 \quad \text{so} \quad z = 0\cdot 4203\,x - 0\cdot 5775\,y. \quad (15)$$

$z = 0$ on the equator and this occurs at $\tan\phi = -v_x/v_y = 0\cdot 7278$, $\phi = 36\cdot 0°$ and $216\cdot 0°$. The axis of rotation of the plane, away from the equator $\theta = \pi/2$, cuts the sphere only 4° away from B.

Though not essential for our purpose, I will comment on the ellipse which is the projection of this great circle onto the $x-y$ plane. Substituting for z in the equation of the sphere gives

$$1\cdot 177x^2 - 0\cdot 489xy + 1\cdot 334y^2 = 1.$$

Now convert this to the standard form of an ellipse rotated in its plane through angle α:

$$\frac{(x\cos\alpha + y\sin\alpha)^2}{a^2} + \frac{(-x\sin\alpha + y\cos\alpha)^2}{b^2} = 1.$$

One semi-axis, a say, must be 1. By matching coefficients one can readily show that $b = 0\cdot 8137$ and $\alpha = 36\cdot 0°$. The inclination of the plane is $\arccos b = 35\cdot 6°$, and indeed we could have found this from the z co-ordinate of $<\hat{\mathbf{c}}\times\hat{\mathbf{b}}>$ which is v_z.

Returning to the calculation of area, the conversion between Cartesian and spherical co-ordinates for points at unit distance from O is

$$x = \sin\theta\cos\phi, \qquad x = \sin\theta\sin\phi, \qquad z = \cos\theta.$$

These satisfy the equation of a unit sphere, $x^2+y^2+z^2 = 1$. Let $\theta_0(\phi)$ define the great circle BC. Substituting into the expression for $z = \cos\theta$ at Eq 15 gives

$$\cos\theta_0 = -\frac{v_x}{v_z}\sin\theta_0\cos\phi - \frac{v_y}{v_z}\sin\theta_0\sin\phi.$$

$$\text{from which} \quad \cot\theta_0 \;=\; -\frac{1}{v_z}(v_x\cos\phi + v_y\sin\phi). \tag{16}$$

In our example $\cot\theta_0 \;=\; 0\cdot 4203\cos\phi - 0\cdot 5775\sin\phi$. θ_0 in Eq 16 is one limit of integration in finding the area of the spherical triangle ABC, and hence of PQR.

An element of area is $\sin\theta\, d\theta\, d\phi$ so the integration is

$$\mathcal{A} \;=\; \int_0^A\int_0^{\theta_0}\sin\theta\, d\theta\, d\phi \;=\; \int_0^A(1-\cos\theta_0)\, d\phi = \;\phi_B - \int_0^A\cos\theta_0\, d\phi \tag{17}$$

A being $40\cdot 0° = 0\cdot 698$ radians in our case. $\cos\theta_0$ can be obtained from the following identity, or otherwise:

$$\cos\theta_0 \;=\; \frac{\cot\theta_0}{\sqrt{1+\cot^2\theta_0}} \;=\; -\frac{-(v_x\cos\phi + v_y\sin\phi)}{\sqrt{v_z^2 + (v_x\cos\phi + v_y\sin\phi)^2}}. \tag{18}$$

The remaining integral is challenging. For the example it can be evaluated numerically and is $0\cdot 1293$. The area of our spherical triangle is then $0\cdot 6981 - 0\cdot 1293 = 0\cdot 5688$ square units, 30% larger than the planar triangle with the same vertices. We have met this value already, near Eq 8b, where it was presented as the 'angular excess'. The challenge is now to show that this is true in general.

To take this forwards combine the cosine and sine terms and introduce a new variable u. Since $R\sin(\alpha+\phi) \;=\; R\sin\alpha\cos\phi + R\cos\alpha\sin\phi$, let $R\sin\alpha = v_x/v_z$ and $R\cos\alpha = v_y/v_z$. Then

$$\cos\theta_0 \;=\; \frac{-R\sin u}{\sqrt{1+R^2\sin^2 u}}, \qquad u \;=\; \phi+\alpha, \qquad R \;=\; \frac{\sqrt{v_x^2+v_y^2}}{v_z}, \quad \tan\alpha \;=\; \frac{v_x}{v_y}.$$

In the integral $d\phi = du$, but the limits change to α and $A+\alpha$. The indefinite integral is given in the handbook by Gradshteyn and Ryzhik (Academic Press) as item 2.597.4, page 173 as

$$\int\frac{\sin u\, du}{\sqrt{1+R^2\sin^2 u}} \;=\; -\frac{1}{R}\arcsin\left(\frac{R\cos u}{\sqrt{1+R^2}}\right).$$

Therefore the area $\mathcal{A}$ is

$$A \;-\; \arcsin\left(\frac{R\cos(A+\alpha)}{\sqrt{1+R^2}}\right) + \arcsin\left(\frac{R\cos\alpha}{\sqrt{1+R^2}}\right), \qquad \frac{R}{\sqrt{1+R^2}} = \sqrt{v_x^2+v_y^2} \tag{19a}$$

since $\sqrt{1+R^2} = 1/v_z$. From $\tan\alpha$ we obtain $\cos\alpha = v_y/\sqrt{v_x^2+v_y^2}$ so

$$\frac{R\cos(A+\alpha)}{\sqrt{1+R^2}} \;=\; v_y\cos A - v_x\sin A, \qquad \frac{R\cos\alpha}{\sqrt{1+R^2}} \;=\; v_y.$$

$$\mathcal{A} \;=\; A - \arcsin(v_y\cos A - v_x\sin A) + \arcsin v_y\,. \tag{19b}$$

In the example the contribution from the lower limit is $0\cdot 4892$, and from the upper is $-0\cdot 6186$ so the area is $2\pi/9 - 0\cdot 6186 + 0\cdot 4892 = 0\cdot 5688$ in agreement with the numerical integration and the proven fact that the area equals the angular excess.

Eq 19b presents the area in quite a different way from the classic 'three overlapping lunes' argument given in a previous section, leaving the question of whether it can be converted into $A+B+C-\pi$. This would mean that $-\arcsin(v_y\cos A - v_x\sin A) + \arcsin v_y = B+C-\pi$. Eq 19 is a function of the unit vector $\hat{\mathbf{v}}$ normal to the yellow cutting plane OBC. Therefore we must relate the components of $\hat{\mathbf{v}}$ to the angles B and C.

The angle between two cutting planes is obtained from the cross product of two vectors, one lying in each plane. Let the unit normal to the blue plane OAC be $\hat{\mathbf{n}} = (0,1,0)$. $\hat{\mathbf{n}}\cdot\hat{\mathbf{v}} = \cos C = v_y$. For $0 < C < \pi$, $\arcsin(\cos C) = \pi/2 - C$, but if $C > \pi/2$, $\cos C$ is negative, making $\arcsin\cos C$ negative. In this case replace $\arcsin(\cos C)$ by $C - \pi/2$.

Let the unit normal to the red plane be $\hat{\mathbf{m}}$.

$$\hat{\mathbf{m}} = \frac{1}{\sqrt{b_x^2+b_y^2}}(-b_y,\, b_x\, 0),$$

Angle B is given by

$$\cos B = \hat{\mathbf{m}}\cdot\hat{\mathbf{v}} = \frac{-b_y v_x + b_x v_y}{\sqrt{b_x^2+b_y^2}} \quad \text{where} \quad \hat{\mathbf{n}}\cdot\hat{\mathbf{m}} = \cos A = \frac{b_x}{\sqrt{b_x^2+b_y^2}}, \quad \text{so} \quad \sin A = \frac{b_y}{\sqrt{b_x^2+b_y^2}},$$

the latter following from $\cos^2 z + \sin^2 z = 1$. It follows that $v_y\cos A - v_x\sin A = \cos B$ and $\arcsin(\cos B) = \pi/2 - B$.

Collecting these results Eq 19b is equal to the angular excess $A+B+C-\pi$.

Closing remarks

The question posed asked us to relate the properties of the planar triangle PQR to those of the spherical triangle which shares the same three vertices. The conclusion is that the two have little else in common. The spherical triangle belongs to a quite different geometry, more akin to projective geometry, in which there is little constraint on the internal angles provided they sum to more than 180°. This makes it more difficult for we humans to visualise and compare areas. The important relations are

1. a strict (authentic) spherical triangle has three edges which are arcs of great circles,
2. its area is equal to the angular excess, $P+Q+R-\pi$, or $A+B+C-\pi$,
3. therefore the triangle with three given angles is unique (except for translation),
4. the Sine rule is
$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c},$$
5. there are two Cosine rules, one the dual of the other:

$$\text{Cosine Rule :} \quad \cos c = \cos a\,\cos b + \sin a\,\sin b\,\cos C.$$

$$\text{Dual Cosine Rule :} \quad \cos C = -\cos A\,\cos B + \sin A\,\sin B\,\cos c.$$

Appendix 1: cone with spherical cap

This appendix considers a cone with spherical base. We find its surface area and volume. The geometry is illustrated in Figure 8. The conical part has apex at O: (0, 0), its length is h along the z axis and the semi-angle is C, making the radius vary as $r = z \tan C$. A slice of cross section at z has surface area $2\pi r\,\delta z$ and volume $\pi r^2\,\delta z$. Integrating, the area A of the conical surface and the volume V contained are

$$A = 2\pi \tan C \int_0^h z.dz = \pi h^2 \tan C, \qquad V = \pi \tan^2 C \int_0^h z^2.dz = \frac{1}{3}\pi h^3 \tan^2 C. \qquad (A1.1)$$

The area of the circular end face is $\pi h^2 \tan^2 C$. V is one third that of the cylinder with these circular ends.

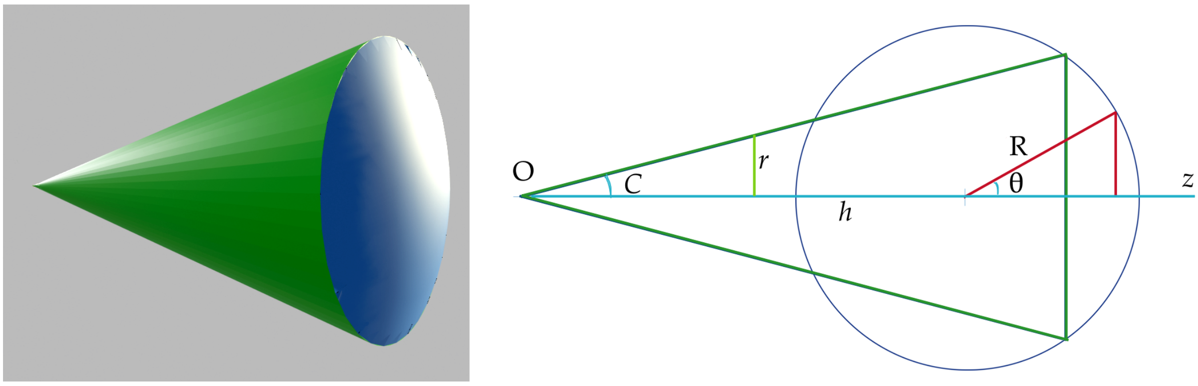

Figure 8: Cone with spherical cap in perspective and in section.

For comparison with the spherical triangle, the spherical end cap on the cone should have radius of curvature such that the apex of the cone is at the centre of the sphere and the cap fits neatly to the circular end. However, it is no more complicated to let the radius of curvature be a more general value R, provided it is not so small ($< h \tan C$) that the spherical cap no longer touches the cone. The object will look somewhat like an ice cream scoop in a cone wafer. There are two complementary arrangements, one with the centre of the sphere inside the cone, the other with the centre outside, in which case the larger part of the ball is also outside and stuck on the end of the cone. Using spherical polar co-ordinates R, θ, ϕ with the pole at $\theta = 0$ along the z axis, an element of surface area of the sphere is $R^2 \sin\theta\, d\theta\, d\phi$. The area of the ring at θ is therefore $2\pi R^2 \sin\theta\, d\theta$. The surface area A_c of the cap to angle θ_0 is

$$A_c = 2\pi R^2 \int_0^{\theta_0} \sin\theta.d\theta = 2\pi R^2 (1 - \cos\theta_0), \qquad (A1.2)$$

while the area of the larger complementary section of the sphere is $2\pi R^2(1 + \cos\theta_0)$. We might choose to express A_c in terms of its depth d from crown to base, which is $R - R\cos\theta_0$. This form is $A_c = 2\pi R d$, a compact expression.

To compare the cap with a spherical triangle we take $R = h/\cos\theta_0$ and $\theta_0 = C$, and choose the small cap option. The area of the cap is then

$$2\pi h^2 \sec C (\sec C - 1). \qquad (A1.3)$$

The ratio of the area of cap to the area of the circular end to the cone is

$$\frac{2}{1+\cos C}. \qquad (A1.4)$$

This function is plotted in Figure 9. It rises from 1 to 2 as the semi-angle of the cone opens to a flat surface and the cap becomes a hemisphere with area $2\pi h^2$ square units.

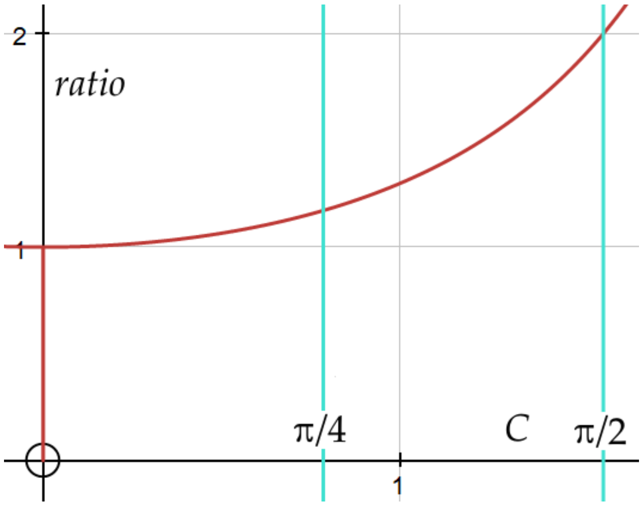

Figure 9: Ratio of areas of the spherical cap to the flat end of the cone as the cone widens.

For completeness, the volume of the spherical cap can be obtained as follows. An equation for the sphere is $x^2+y^2+(z+R-d)^2=R^2$, d again being the depth of the cap. The axial symmetry allows us to write $x^2+y^2=\rho^2$. The depth of an element of volume at radial distance ρ from the z axis is $z=\sqrt{R^2-\rho^2}-R+d$. If the circular base of the cap is divided into thin concentric rings of width $d\rho$, the area of the ring at ρ is $2\pi\rho\, d\rho$ and the volume above it, from base to outer surface, is $2\pi\rho\, z(\rho)\, d\rho$. The volume of the whole cap is the integral of this out to $\rho = R\sin\theta_0$:

$$V_c = 2\pi\int_0^{R\sin\theta_0}\rho(\sqrt{R^2-\rho^2}-R+d)\,d\rho.$$

This evaluates to

$$V_c = \frac{\pi R^3}{3}(2+\cos\theta_0)(1-\cos\theta_0)^2. \qquad (A1.5)$$

For $\theta_0=\pi/2$ the solid is a hemisphere with volume $2\pi R^3/3 = 2{\cdot}094R^3$. If $\theta_0=\pi/4$, the enclosed volume is only $0\cdot234R^3$, less than 1/8 of the hemisphere. On Earth latitude 45° north runs through the Pyrenees, Venice, Crimea, Japan, Oregon in the USA and Quebec.

Appendix 2: Vector proofs of Sine and Cosine Rules

Sine Rule

The proof of the Sine rule for spherical triangles stated at Eq 10a, b is as follows. Bear in mind that $\hat{\mathbf{p}}$, $\hat{\mathbf{q}}$, $\hat{\mathbf{r}}$ are unit vectors. The aim is to show that

$$\frac{\sin P}{\sin\eta} = \frac{\sin Q}{\sin\theta}.$$

$$\sin P = < (\hat{\mathbf{p}}\times\hat{\mathbf{r}})\times\hat{\mathbf{p}}\times\hat{\mathbf{q}} > = \left|\frac{\hat{\mathbf{p}}\times\hat{\mathbf{r}}}{|\hat{\mathbf{p}}\times\hat{\mathbf{r}}|}\times\frac{\hat{\mathbf{p}}\times\hat{\mathbf{q}}}{|\hat{\mathbf{p}}\times\hat{\mathbf{q}}|}\right|,$$

$$\sin Q = < (\hat{\mathbf{q}}\times\hat{\mathbf{p}})\times\hat{\mathbf{q}}\times\hat{\mathbf{r}} > = \left|\frac{\hat{\mathbf{q}}\times\hat{\mathbf{p}}}{|\hat{\mathbf{q}}\times\hat{\mathbf{p}}|}\times\frac{\hat{\mathbf{q}}\times\hat{\mathbf{r}}}{|\hat{\mathbf{q}}\times\hat{\mathbf{r}}|}\right|.$$

Take the sines of the edge arc lengths:

$$\text{arc QR:}\quad \sin\eta = |\hat{\mathbf{q}}\times\hat{\mathbf{r}}|, \qquad \text{arc PR:}\quad \sin\theta = |\hat{\mathbf{p}}\times\hat{\mathbf{r}}|.$$

Cross multiply. The two sides of the conjectured equality are

$$\text{Left :}\quad \left|\frac{\hat{\mathbf{p}}\times\hat{\mathbf{r}}}{|\hat{\mathbf{p}}\times\hat{\mathbf{r}}|}\times\frac{\hat{\mathbf{p}}\times\hat{\mathbf{q}}}{|\hat{\mathbf{p}}\times\hat{\mathbf{q}}|}\right||\hat{\mathbf{p}}\times\hat{\mathbf{r}}|, \qquad \text{Right :}\quad \left|\frac{\hat{\mathbf{q}}\times\hat{\mathbf{p}}}{|\hat{\mathbf{q}}\times\hat{\mathbf{p}}|}\times\frac{\hat{\mathbf{q}}\times\hat{\mathbf{r}}}{|\hat{\mathbf{q}}\times\hat{\mathbf{r}}|}\right||\hat{\mathbf{q}}\times\hat{\mathbf{r}}.$$

Cancel equal factors in numerator and denominator:

$$\text{Left :}\quad \left|(\hat{\mathbf{p}}\times\hat{\mathbf{r}})\times\frac{\hat{\mathbf{p}}\times\hat{\mathbf{q}}}{|\hat{\mathbf{p}}\times\hat{\mathbf{q}}|}\right||, \qquad \text{Right :}\quad \left|\frac{\hat{\mathbf{q}}\times\hat{\mathbf{p}}}{|\hat{\mathbf{q}}\times\hat{\mathbf{p}}|}\times(\hat{\mathbf{q}}\times\hat{\mathbf{r}})\right|.$$

The norms in the denominators are equal so the equality to be proved is

$$\text{Left :}\quad |(\hat{\mathbf{p}}\times\hat{\mathbf{r}})\times(\hat{\mathbf{p}}\times\hat{\mathbf{q}})|, \qquad \text{Right :}\quad |(\hat{\mathbf{q}}\times\hat{\mathbf{p}})\times(\hat{\mathbf{q}}\times\hat{\mathbf{r}})|.$$

At this stage we need two well established vector identities for general vectors $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$:

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(A\cdot C)-\mathbf{C}(A\cdot B), \qquad (\mathbf{A}\times\mathbf{B})\times\mathbf{C} = \mathbf{B}(A\cdot C)-\mathbf{A}(B\cdot C), \qquad (A2.1)$$

the dot products just being scalars. First let $\mathbf{A}=\hat{\mathbf{p}}\times\hat{\mathbf{r}}$, $\mathbf{B}=\hat{\mathbf{p}}$, $\mathbf{C}=\hat{\mathbf{q}}$ to obtain

$$(\hat{\mathbf{q}}\times\hat{\mathbf{p}})\times(\hat{\mathbf{q}}\times\hat{\mathbf{r}}) = \hat{\mathbf{p}}(\hat{\mathbf{p}}\times\hat{\mathbf{r}}\cdot\hat{\mathbf{q}}) - \hat{\mathbf{q}}(\hat{\mathbf{p}}\times\hat{\mathbf{r}}\cdot\hat{\mathbf{p}}) = \hat{\mathbf{p}}(\hat{\mathbf{p}}\times\hat{\mathbf{r}}\cdot\hat{\mathbf{q}}), \qquad (A2.2)$$

the second term being zero. The triple product is a scalar, equal to the volume of the parallelepiped spanned by the three vectors. Since $\hat{\mathbf{p}}$ is a unit vector, the norm of the left side is simply $\hat{\mathbf{p}}\times\hat{\mathbf{r}}\cdot\hat{\mathbf{q}}$. Now we apply the second identity to the right side, letting $\hat{\mathbf{A}}=\hat{\mathbf{q}}$, $\hat{\mathbf{B}}=\hat{\mathbf{p}}$ and $\hat{\mathbf{C}}=\hat{\mathbf{q}}\times\hat{\mathbf{r}}$.

$$\hat{\mathbf{p}}(\hat{\mathbf{q}}\cdot\hat{\mathbf{q}}\times\hat{\mathbf{r}}) - \hat{\mathbf{q}}(\hat{\mathbf{p}}\cdot\hat{\mathbf{q}}\times\hat{\mathbf{r}}) = -\hat{\mathbf{q}}(\hat{\mathbf{p}}\cdot\hat{\mathbf{q}}\times\hat{\mathbf{r}}).$$

Again $\hat{\mathbf{q}}$ is a unit vector so the norm of the right side is also $\hat{\mathbf{p}}\cdot\hat{\mathbf{q}}\times\hat{\mathbf{r}}$. That completes the proof.

For the example the triple product is $0\cdot5915$ and

$$\frac{\sin P}{\sin\eta} = \frac{\sin Q}{\sin\theta} = \frac{\sin R}{\sin\xi} = \frac{\hat{\mathbf{p}}\cdot(\hat{\mathbf{q}}\times\hat{\mathbf{r}})}{|\hat{\mathbf{p}}\times\hat{\mathbf{q}}||\hat{\mathbf{q}}\times\hat{\mathbf{r}}||\hat{\mathbf{r}}\times\hat{\mathbf{p}}|}, \qquad (A2.3)$$

a constant property of the triangle.

Cosine Rule

Translate Eq 11a into vectors:

$$\begin{aligned}
\cos\eta - \cos\theta\cos\xi &= \sin\theta\sin\xi\cos P \\
\hat{\mathbf{q}}\cdot\hat{\mathbf{r}} - (\hat{\mathbf{p}}\cdot\hat{\mathbf{q}})(\hat{\mathbf{p}}\cdot\hat{\mathbf{r}}) &= |\hat{\mathbf{p}}\times\hat{\mathbf{r}}|\,|\hat{\mathbf{p}}\times\hat{\mathbf{q}}|\left(\frac{\hat{\mathbf{p}}\times\hat{\mathbf{r}}}{|\hat{\mathbf{p}}\times\hat{\mathbf{r}}|}\cdot\frac{\hat{\mathbf{p}}\times\hat{\mathbf{q}}}{|\hat{\mathbf{p}}\times\hat{\mathbf{q}}|}\right) \\
&= (\hat{\mathbf{p}}\times\hat{\mathbf{r}})\cdot(\hat{\mathbf{p}}\times\hat{\mathbf{q}}).
\end{aligned}$$

The right side is a relabelling of Eq A2.2.

$$(\hat{\mathbf{p}}\times\hat{\mathbf{r}})\cdot(\hat{\mathbf{p}}\times\hat{\mathbf{q}}) = (\hat{\mathbf{p}}\cdot\hat{\mathbf{p}})(\hat{\mathbf{r}}\cdot\hat{\mathbf{q}}) - (\hat{\mathbf{p}}\cdot\hat{\mathbf{q}})(\hat{\mathbf{r}}\cdot\hat{\mathbf{p}})$$

and since $\hat{\mathbf{p}}\cdot\hat{\mathbf{p}} = 1$, this is the left hand side.

Appendix 3: Rotation matrix

This pseudo-code calculates the matrix elements for rotating three points P, Q, R on a sphere to standard position with P → A: (0,0,1), Q → B: (b_x, b_y, b_z), R → C: $(c_x, 0, c_z)$.

```
numa11 = py^2*rx - px*py*ry + pz*(pz*rx - px*rz)
     dena11 = SQRT( (px^2 + py^2 + pz^2)* ( pz^2*(rx^2 + ry^2)
     - 2*px*pz*rx*rz - 2*py*ry*(px*rx + pz*rz) + py^2*(rx^2 + rz^2)
     + px^2*(ry^2 + rz^2) ) )
     a11 = numa11/dena11
     numa21 = -(pz*ry - py*rz)*( (py^2 + pz^2)*rx - px*py*ry - px*pz*rz)
     dena21 = a11*( (py^2 + pz^2)*rx^2 - 2*px*py*rx*ry +
     (px^2 + pz^2)*ry^2 - 2*px*pz*rx*rz - 2*py*pz*ry*rz
     + (px^2 + py^2)*rz^2)
     a21 = numa21/dena21
     a32 = py/(px^2 + py^2 + pz^2)
     a31 = (-px*pz*rx - (py*(-1 + a32*py) + a32*pz^2) * (-pz*ry + py*rz)
     + px^2*(a32*pz*ry + rz - a32*py*rz))
     / ((px^2 + py^2 + pz^2)*(-pz*rx + px*rz))
     numa12 = -px*py*rx + px^2*ry + pz*(pz*ry - py*rz)
     dena12 = py^2*rx - px*py*ry + pz*(pz*rx - px*rz)
     a12 = a11* numa12/dena12
     a22 = a21*(-pz*rx + px*rz)/(pz*ry - py*rz)
     a33 = (1 - a31*px - a32*py)/pz
     a23 = (-a21*px - a22*py)/pz
     a13 = (-a11*px - a12*py)/pz


     PRINT" Transformed vectors are :  "
     ppx = a11*px + a12*py + a13*pz
     ppy = a21*px + a22*py + a23*pz
     ppz = a31*px + a32*py + a33*pz

     qqx = a11*qx + a12*qy + a13*qz
     qqy = a21*qx + a22*qy + a23*qz
     qqz = a31*qx + a32*qy + a33*qz
     rrx = a11*rx + a12*ry + a13*rz
     rry = a21*rx + a22*ry + a23*rz
     rrz = a31*rx + a32*ry + a33*rz
```