Series of Pythagorean Triples (3, 4, 5), (5, 12, 13), etc.

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1 Set of three integers with $a^2 + b^2 = c^2$

It has been known for centuries that there are very many sets of three integers with this property. The statement $a^2 + b^2 = c^2$ can be represented geometrically as the lengths of the sides of a right-angled triangle, with c being the hypotenuse. This is Pythagoras's theorem, hence the name 'Pythagorean triples'. I mention in passing the famous Fermat's Last Theorem which states that $a^n + b^n = c^n$ can be true in integers only if $n = 2$. As is well known, this was eventually proved by Andrew Wiles in 1994.

It has also been known for centuries that there is a formula for calculating all Pythagorean triples. It appears in textbooks on basic number theory. This article is my personal exploration, without looking up the formula in a book. I have noticed that there are sequences of such triples.

To start, let us consider the parity (odd or even) of the integers a, b, c. First if c is even, c^2 is divisible by 4. This means that $a^2 + b^2$ is divisible by 4, which can happen only if both a and b are even – otherwise $a^2 + b^2 \mod 4 = 1$ or 2, not 0. This case, therefore, is a multiple of a simpler case in which a, b, c have no common factors.

It is also easy to see that a and b cannot both be odd, since then $a^2 + b^2 \mod 4 = 2$, while a squared number c^2 must be either $0 \mod 4$ or $1 \mod 4$. We conclude that a is even, and b and c are both odd. The two examples given in the title of this article, (3, 4, 5) and (5, 12, 13), conform to this. So write

$$a = 2p, \qquad b = 2q + 1, \qquad c = 2r + 1, \tag{1}$$

where p, q, r are also integers[1]. With these

$$\begin{aligned} 4p^2 + 4q^2 + 4q + 1 &= 4r^2 + 4r + 1 \\ p^2 + q^2 + q &= r^2 + r. \end{aligned} \tag{2}$$

We will be working mostly with Eq 2.

2 One simple series

This series is obtained from Eq 2 by setting $r = p$. Then

$$p = q^2 + q = q(q + 1). \tag{3}$$

[1] In principle p, q, r could be odd multiples of 1/2 and still leave a, b, c as integers. However c would then be even, and the triple a multiple of a coprime triple.

The integers $q \geq 1$ generate a countably infinite series of Pythagorean triples. Here are the first few. I have ordered each triple in ascending size:

q	(q,p,r)	(b,a,c)	q	(q,p,r)	(b,a,c)
1	$(1,2,2)$	$(3,4,5)$	7	$(7,56,56)$	$(15,112,113)$
2	$(2,6,6)$	$(5,12,13)$	8	$(8,72,72)$	$(17,144,145)$
3	$(3,12,12)$	$(7,24,25)$	9	$(9,90,90)$	$(19,180,181)$
4	$(4,20,20)$	$(9,40,41)$	10	$(10,110,110)$	$(21,220,221)$
5	$(5,30,30)$	$(11,60,61)$	11	$(11,132,132)$	$(23,264,265)$
6	$(6,42,42)$	$(13,84,85)$	12	$(12,156,156)$	$(25,312,313)$

The series is characterised by the two largest integers, a and c, differing by only 1. Note that each triple (a, b, c) has no common factors.

3 Further series

Another way of looking at Eq 2 is

$$p^2 = r^2 + r - q^2 - q = (r^2 - q^2) + (r - q)$$

$$p^2 = (r+q)(r-q) + (r-q) = (r+q+1)(r-q). \tag{4}$$

This casts p^2 as the product of two unequal factors, and implies that, for example, for $p = 4$, $p^2 = 16$, the factor pairs (1, 16) and (2, 8) should each generate q and r values leading to a Pythogorean triple. If the factors of p^2 are $j = r - q$, $k = r + q + 1$, their simultaneous solution is

$$q = \tfrac{1}{2}(k - j - 1), \qquad r = \tfrac{1}{2}(k + j - 1)$$

$$\text{from which} \quad b = k - j, \quad c = k + j. \tag{5}$$

A particular case is $j = 1$, $k = p^2$ giving $r = p^2/2$, $q = p^2/2 - 1$, $a = 2p$, $b = p^2 - 1$, $c = p^2 + 1$. We have already recognised (see footnote on page 1) that if any of p, q, r is 'integer + 1/2', the Pythagorean triple will be a multiple of a simpler one whose integers are coprime. We therefore restrict to p even. Here is a list of cases, again generally in order of increasing size..

p	(p,q,r)	(a,b,c)	p	(p,q,r)	(a,b,c)
2	$(2,1,2)$	$(4,3,5)$	14	$(14,97,98)$	$(28,195,197)$
4	$(4,7,8)$	$(8,15,17)$	16	$(16,127,128)$	$(32,255,257)$
6	$(6,17,18)$	$(12,35,37)$	18	$(18,161,162)$	$(36,323,325)$
8	$(8,31,32)$	$(16,63,65)$	20	$(20,199,200)$	$(40,399,401)$
10	$(10,49,50)$	$(20,99,101)$	22	$(22,241,242)$	$(44,483,485)$
12	$(12,71,72)$	$(24,143,145)$	24	$(24,287,288)$	$(48,575,577)$
p		$(2p,\ p^2-1,\ p^2+1)$			

So here is another infinite sequence of Pythagorean triples, this time with the two largest numbers differing by 2. If p is odd, the triples generated are (6, 8, 10), (10, 24, 26), (14, 48, 25) *etc.* which are simply doubles of the one found in Section 2.

We move on to consider the non-trivial factorisations of p^2. Eq 5 states that the Pythagorean triple will be $a = 2p$, $b = p^2 - j$, $c = p^2 + j$ where j is a factor of p^2. To avoid generating mere multiples of co-prime triples, two conditions must be satisfied:

1. c must be odd, which requires that only one of j and k be even,
2. factors j and k must be coprime, else their common factor would carry forwards into $b = k - j$, $c = k + j$.

Condition 1 is a special case of condition 2. They mean that j and k must each be a square, one odd, one even. So call $j = u^2$ and $k = v^2$. Eq 5 leads to

$$a = 2p = 2\sqrt{jk} = 2uv,$$

$$b = k - j = v^2 - u^2, \qquad c = k + j = v^2 + u^2.$$

The table below lists the first few integers p, the coprime factors of p^2 and the Pythagorean triples they create.

p	p^2	$j = \sqrt{u}$	$k = \sqrt{v}$	a	b	c
6	36	4	9	12	5	13
10	100	4	25	20	21	29
12	144	9	16	24	7	25
14	196	4	49	28	45	53
18	324	4	81	36	77	85
20	400	16	25	40	9	41
22	484	4	121	44	117	125
24	576	9	64	48	55	73
26	676	4	169	52	165	173
28	784	16	49	56	33	65
30	900	9	100	60	91	109
30	900	25	36	60	11	61
34	1156	4	289	68	285	293
36	1296	9	144	72	135	153
36	1296	16	81	72	65	97
38	1444	4	361	76	357	365
40	1600	25	64	80	39	89

Some of these triples were listed in the table of §2, but more are new. This list is the start of a third infinite sequence of Pythagorean triples. It appears that sequence 1 with $r = p$, §2, is duplicated in this third sequence, and indeed they correspond to $v - u = 1$. Together these three sequences seem to be exhaustive.

The tables show 17 Pythagorean triples for which c is 101 or less; the c values are

$$5,\ 13,\ 17,\ 25,\ 29,\ 37,\ 41,\ 53,$$

$$61,\ 65,\ 65,\ 73,\ 85,\ 85,\ 89,\ 97,\ 101.$$

Of these 12 are prime and two (65 and 85) occur twice.

The title of this article mentions series of Pythagorean triples. In addition to the series whose highest two integers differ by 1 and by 2, we could pick out from the above list ones wich differ by 8, 18 or any value $2u^2$ for integer u.

4 Comment

So we have found three infinite sequences of Pythagorean triples and formulae for generating them. 'So what?' we may ask. Have they any significance beyond being numerical coincidences? I know of only one other area of maths or science where they appear, and that is giving the points on the circle

$$\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1,$$

with axes a and b, which have fractional (rational) co-ordinates. But this itself is perhaps just a curiosity. For instance, I have seen no suggestion that such points form a group, which would make them interesting. Moreover, the angles of the triangles whose side lengths have triple integer values seem to have no special significance. The ratios formed by the integers of each triple (sines, cosines, tangents) do not seem related to other notable fractional numbers. I therefore see no reason to view them as any more than mathematical coincidences.

Nevertheless, they have fascinated people since Euclid wrote his text on them. Perhaps it is the formula for finding them which is intriguing. So apart from practical help to the ancient Greeks in contructing right angles, perhaps their most important role has been to stir interest in numbers and in particular to prompt Fermat's Last Theorem. They may also have motivated the search for sums of three cubed integers which equal a fourth cube, the sums of four squares, some Diophantine equations, and more, such is the addictive nature of number theory.