Dynamics of a Mass Revolving on an Elastic Chord

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Lagrange equations, centrifugal force, coupled oscillations, conservation of momentum

1 The Dynamical System

I came across this problem in dynamics when considering the even more complicated elastic pendulum problem. Imagine a bob of mass m attached to a spring or elastic chord, natural length h_0 and stiffness k. The chord is held fixed at its other end, O, and the mass revolves round O with the chord taut. The mass is supported on a smooth, lubricated horizontal table where friction can be neglected. At some time $t = 0$ the mass is given an impulsive outwards pull, making the chord instantaneously extend further. Determine the subsequent motion of the mass as a function time and of the parameters.

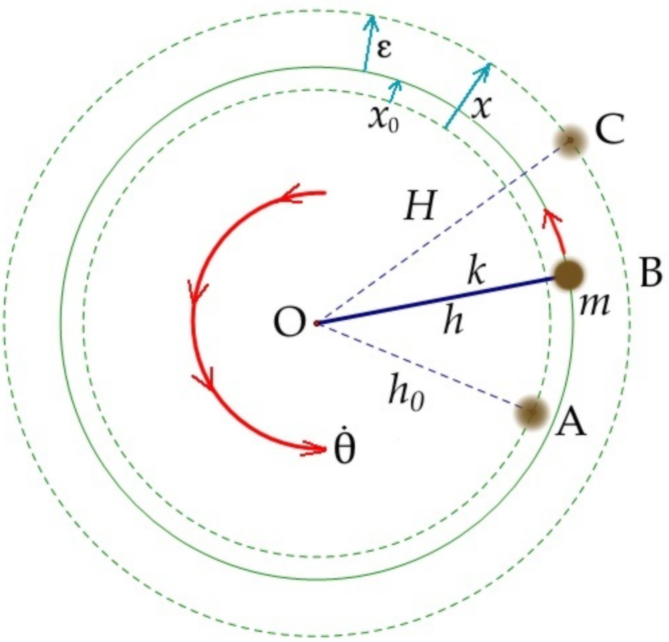

Figure 1: Mass m on horizontal table revolving at rate $\dot{\theta}$ about centre O, held on elastic chord.

The system is illustrated in Figure 1 which shows the bob at three radial positions corresponding to three stages of its history. At A, the initial condition, it is stationary with the chord at its unstretched length h_0. The spring is now caused to turn by a torque, and spins up to a steady angular velocity[1] $\dot\theta_0$. By B the torque has been removed and the bob is revolving in constant circular motion. There is now tension in the chord needed to restrain it in a circle and under this tension the chord extends an extra length x_0 to h. If the angular velocity in this steady revolution is $\dot\theta_0$, the bob seems to experience an outwards centrifugal force $mh\dot\theta_0^{\,2}$ which is balanced by the tension in the spring.

$$h \;=\; h_0 + x_0 \;=\; h_0 \;+\; \frac{mh}{k}\dot\theta_0^{\,2} \quad \text{so} \quad h \;=\; \frac{h_0}{1-\frac{m}{k}\dot\theta_0^{\,2}} \quad \text{and} \quad x_0 \;=\; \frac{h_0\mu}{1-\mu} \tag{1}$$

where $\mu = m\dot\theta_0^{\,2}/k$, a quantity of steady revolution which will feature in the analysis below.

Some time later, at position C the bob is pulled outwards quickly until the total extension in the spring is x. If the bob were make of iron, this might be achieved by an impulse of attraction from a strong magnet at C as it passed by. The instantaneous chord length is $H(t) = h_0 + x$ so the tension is kx. The centrifugal force has increased to $mH\dot\theta^2$, but there is an excess elastic force which causes inwards radial acceleration and thence oscillation. However the problem is complicated by the fact that $\dot\theta \neq \dot\theta_0$. This is because the angular momentum G of the whirling mass stays constant, so $\dot\theta$ depends on H. Now $G = I\dot\theta$ where $I = mH^2$ is the moment of inertia. Take the reference state to be as at B when the angular velocity was $\dot\theta_0$ and $H = h$. Then

$$G \;=\; mh^2\dot\theta_0 \;=\; mH^2\dot\theta \;\equiv\; m\{h_0 + x(t)\}^2\dot\theta(t)\,. \tag{2}$$

The constant angular momentum imposes a coupling between the instantaneous radial position of the bob and its instantaneous angular velocity.

2 Equations of motion

I have had two lines of thought on how to derive the equations of motion; it will be interesting to see whether they agree.

2.1 Forces in a rotating frame

Though the system exists in two space dimensions, the angle θ itself does not appear in the above formulae, only the angular velocity $\dot\theta$. The problem can therefore be considered as one-dimensional, in the radial direction only. The effect of the circular motion is included by invoking the so-called fictitious forces which arise when an inertial frame of reference is replaced by a rotating frame[2]. The three fictitious forces are

1. Euler force $m\,\ddot{\boldsymbol\theta}\times\mathbf{H}$, due to rotational acceleration of the frame. The $\times$ sign denotes vector cross product. Since $\ddot{\boldsymbol\theta}$ is along the axis of rotation, normal to the page in Figure 1, the Euler force is in the circumferential, θ, direction.

2. Coriolis force $2m\,\dot{\boldsymbol\theta}\times\dot{\mathbf{H}}$. This exists only when the bob is moving radially with velocity $\dot H$. This too points circumferentially.

3. Centrifugal force $m\,\dot{\boldsymbol\theta}\times(\dot{\boldsymbol\theta}\times\mathbf{H})$. This points in the radial direction.

The apparent forces therefore separate into radial ones – elastic and centrifugal – and circumferential ones – Euler and Coriolis – giving two equations. Newton's second law gives the radial equation

$$-k(H-h_0) + mH\dot\theta^2 \;=\; m\ddot H \tag{3a}$$

[1] The dot $\dot{}$ means differentiation with respect to time.
[2] See, for example, page 475 in Douglas Gregory's book 'Classical Mechanics' CUP, 2006.

Since within the rotating frame all motion of the bob appears radial, the circumferential equation is

$$H\ddot{\theta} + 2\dot{\theta}\,\dot{H} \;=\; 0\,. \tag{3b}$$

These two equations decouple and their meanings become clearer when the conservation of angular momentum in Eq 2 is made explicit. First express the centrifugal force in terms of G

$$mH\dot{\theta}^2 \;=\; \frac{G^2}{mH^3}\,.$$

Then Eq 3a becomes

$$-k(H - h_0) + \frac{G^2}{mH^3} \;=\; m\ddot{H}\,, \tag{4a}$$

$$\text{equivalent to} \qquad m\ddot{x} + kx - \frac{G^2}{m(h_0 + x)^3} \;=\; 0\,. \tag{4b}$$

$$\text{and to} \qquad m\ddot{\epsilon} + k(x_0 + \epsilon) - \frac{G^2}{m(h + \epsilon)^3} \;=\; 0\,. \tag{4c}$$

x_0 and ϵ are defined respectively by the extensions $h - h_0$ and $H - h = H - x_0 - h_0$ in Figure 1. The variants of this equation are all in only one variable which quantifies the extension. Second differentiate Eq 2 with respect to time.

$$mH^2\dot{\theta} = G,\ \text{constant, so}\ H\ddot{\theta} + 2\dot{\theta}\dot{H} \;=\; 0$$

which is Eq 3b. Here is an important physical insight; the zero net apparent circumferential force on the rotating mass is due to the conservation of angular momentum.

2.2 Lagrange's equations

This line of argument deals with the problem in two dimensions. Neither the non-inertial centrifugal force nor the conservation of angular momentum are included explicitly though both emerge from the formulation. Lagrange's method[3] is a protocol for determining the equations of motion. In our case the elastic force is conservative and can be represented by a potential energy, the gradient of which gives the tension in the spring. If $q_j,\ j = 1, 2$ are the generalised co-ordinates and T and V are respectively the total kinetic and potential energies, the equations of motion are

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} + \frac{\partial V}{\partial q_j} \;=\; 0\,. \tag{5}$$

The co-ordinate derivatives here have a peculiar interpretation; they are performed as if q and $\dot{q}$ were independent variables. In our case $q_1 = x(t)$ and $q_2 = \theta(t)$. Moreover, though we know that angular velocity $\dot{\theta}$ depends on the extension x, in the Lagrange device this is ignored and they are treated as independent.

The potential energy V is all strain energy in the stretched or compressed spring:

$$V \;=\; \tfrac{1}{2}kx^2 \quad \text{so the required derivatives of } V \text{ are} \quad \frac{\partial V}{\partial x} \;=\; kx\,, \qquad \frac{\partial V}{\partial \theta} \;=\; 0\,. \tag{6}$$

To find the kinetic energy the velocity components in the radial and circumferential directions must be added vectorally.

$$T \;=\; \tfrac{1}{2}m(\dot{x}^2 + H^2\dot{\theta}^2)\,, \qquad H = h_0 + x\,. \tag{7}$$

[3] See for instance the book by Douglas Gregory, 'Classical Mechanics', CUP, page 339.

The derivative with respect to x works out to be the centrifugal force:

$$\frac{\partial T}{\partial x} = mH\dot{\theta}^2, \qquad \frac{\partial T}{\partial \theta} = 0. \tag{8}$$

The derivative with respect to $\dot{x}$ is the radial momentum and to $\dot{\theta}$ is the angular momentum:

$$\frac{\partial T}{\partial \dot{x}} = m\dot{x}, \qquad \frac{\partial T}{\partial \dot{\theta}} = mH^2\dot{\theta}.$$

We now need the time derivatives of these two.

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) = m\ddot{x}, \qquad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) = mH^2\ddot{\theta} + 2mH\dot{\theta}\dot{x}.$$

The equations of motion are

$$m\ddot{x} - mH(x)\dot{\theta}^2 + kx = 0. \tag{9a}$$

$$mH[H\ddot{\theta} + 2\dot{\theta}\dot{x}] = 0. \tag{9b}$$

This second equation expresses the time invariance and hence conservation of angular momentum. It justifies us in writing $mH^2\dot{\theta} = G$, a constant. From this we find that $\ddot{\theta}^2 = G^2/(m^2H^4)$. Substituting this into Eq 9a eliminates $\dot{\theta}$ and gives the equation of radial motion found by less rigorous means as Eq 4.

3 Solution for small amplitude oscillations

Eq 4b depends only on x. Clearly it is non-linear and I could not guess a form for the solution. The symbolic software Mathematica 10 is able to obtain an answer, but it a grotesque function involving elliptic integrals. I therefore propose to consider it as a series in the chord's extension, assuming the perturbation from the steady circular motion is small.

The base case is when the system is stationary. Assuming the elastic material to be a spring which can support compression, the equation is motion reduces to

$$-k(H - h_0) = m\ddot{H} \quad \text{or} \quad m\ddot{x} + kx = 0 \tag{10a}$$

$$\text{with solution} \quad H(t) = h_0 + C_1\cos\omega_0 t + C_2\sin\omega_0 t, \quad \omega_0^2 = \frac{k}{m}. \tag{10b}$$

This is simple harmonic motion about the unstretched length h_0.

When the bob is spinning without oscillating radially, the steady extension is $x_0 = h - h_0 = \mu h_0/(1-\mu)$, $\mu = mh\dot{\theta}^2$. Let $\epsilon(t) = H(t) - h$ be the radial distance relative to this stable extension. Then $H(t) = h_0 + x(t) = h + \epsilon(t)$. Recall that in terms of ϵ the equation of radial motion, Eq 4, is

$$m\ddot{\epsilon} + k(h - h_0 + \epsilon) - \frac{G^2}{m(h+\epsilon)^3} = 0. \tag{11}$$

An approximate equation of motion is obtained by expanding $1/H^3$ in powers of $\epsilon(t)$:

$$\frac{1}{H^3} = \frac{1}{(h+\epsilon)^3} = \frac{1}{h^3} - \frac{3\epsilon}{h^4} + \frac{6\epsilon^2}{h^5} - \frac{10\epsilon^2}{h^6} + \ldots\ldots \tag{12}$$

Retaining only the linear term, the equation of motion for small amplitude oscillations is

$$m\ddot{\epsilon} + \left(k + \frac{3G^2}{mh^4}\right)\epsilon \;=\; \frac{G^2}{mh^3} - kx_0 \;=\; 0\,. \tag{13}$$

It is heartening to see that the right hand side simplifies to 0. This equation describes simple harmonic motion $\epsilon = C_1\cos\omega_1 t + C_2\sin\omega_1 t$ with frequency ω_1 given by

$$\omega_1^2 \;=\; \frac{k}{m} + \frac{3G^2}{m^2h^4} \;=\; \frac{k}{m} + 3\dot{\theta_0}^2\,. \tag{14}$$

Spinning the bob therefore increases the frequency of radial oscillation because the effective stiffness constant is increased due to the revolving motion; the increase is $3m\dot{\theta_0}^2$, a value proportional to the centrifugal force.

The next point to assess is how $\dot{\theta}(t)$ varies from the constant $\dot{\theta}_0$. We can expect it to vary at the same frequency as the extension and in phase. From Eq 2 for G

$$\dot{\theta}(t) \;=\; \left(\frac{h}{H(t)}\right)^2\dot{\theta}_0 \;=\; \left(\frac{h}{h+\epsilon(t)}\right)^2\dot{\theta}_0\,.$$

For small amplitude oscillations

$$\dot{\theta} \;\approx\; \dot{\theta}_0\left[1 \;-\; \frac{2\epsilon}{h} \;+\; \frac{3\epsilon^2}{h^2} \;+\; \;....\right], \qquad h \;=\; \frac{h_0}{1-\frac{m}{k}\,\dot{\theta_0}^2}\,. \tag{15}$$

4 Numerical examples

The software Mathematica 10 can carry out numerical integration of the exact radial equation of motion Eqs 4c, 11. A few numerical examples will allow us to assess the range of validity of the linear approximation for which an analytical solution can be found.

Whilst it is possible mathematically to put almost any values into the expressions and equation, in any practical case the parameters will be limited by material reality. For instance, Eq 1 implies that the spring could in principle extend to infinite length as $\mu \to 1$. Of course any real spring would first deform plastically then break long before this mathematical limit were reached. In practice the extension might be limited to 50%, which would limit μ to $\frac{1}{3}$. In SI units, if $m = 1$ kg and $k = 20$ Pa/m, the maximum $\dot{\theta}$ would then be about $2\cdot 6$ radians/sec, at which the bob is moving at $2\cdot 6$ m/sec. Similarly the displacement amplitude ϵ should be limited to about $0\cdot 2$ m.

The three important aspects to check are whether

1. the displacement ϵ varies in a sinusoidal manner,
2. the frequency increases with $\dot{\theta}$ as Eq 14.
3. the central position of radial oscillation is $\epsilon = 0$.

All the graphs plotted by Mathematica in numerical solution have a clear sinusoidal shape with no obvious deformation. Regarding point 2, Figure 2 plots the frequency squared against angular velocity for small radial displacement and shows a parabola fitting closely to Eq 14 up to $\theta_0 = 4$. This is far and away the largest effect of the rotation. There is a small effect of displacement

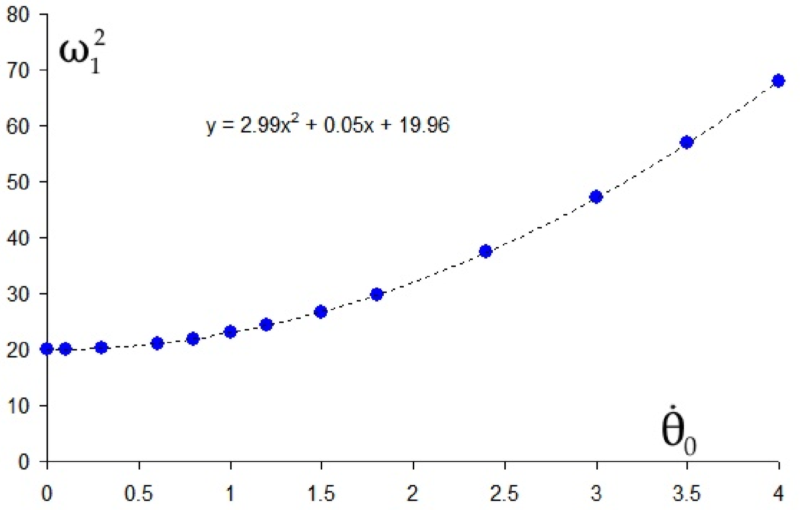

Figure 2: Plot from numerical solutions giving ω_1^2 against θ_0 for $k = 20$, $m = 1$, $h_0 = 1$, $\epsilon(t = 0) = 0.01$ showing close fit to $k/m + 3\dot{\theta}^2$ up to $\theta_0 = 4$.

ϵ on the oscillation frequency. This is greatest between about $\theta_0 = 1 \cdot 2$ and $2 \cdot 5$. For instance, at $\theta_0 = 1 \cdot 8$, $\omega_1 = 5 \cdot 46$ at $\epsilon = 0 \cdot 01$, rising to $5 \cdot 59$ at $\epsilon = 0 \cdot 4$.

Regarding point 3, the sinusoidal oscillation is not symmetrical about $\epsilon = 0$ except for ϵ very small. The numerical evidence is that the outward limit of oscillation remains at $\epsilon = \epsilon(0) > 0$, its starting value, but the inwards half of the cycle becomes shorter. Thus the peak-to-trough distance decreases and the mean position of the oscillation creeps outwards by a small amount. I find that the maximum inwards displacement is approximately $-0 \cdot 87$ of the outwards displacement for $\epsilon > 0 \cdot 01$, more or less irrespective of the angular velocity.

This numerical evidence might suggest a refined formula for $\epsilon(t)$ which would give a more accurate solution of the radial equation of motion, though the numerical evidence is that the linear approximation of §3 captures the essence.

John Coffey, Cheshire, England, December 2015, with grateful thanks to Prof. Douglas Gregory for his patient willingness to discuss this and other problems in mechanics.