Q10 : Evaluate

$$\sqrt[8]{2207 - \cfrac{1}{2207 - \cfrac{1}{2207- \ ...}}} \quad .$$

Express your answer in the form $\frac{a+b\sqrt{c}}{d}$ where a, b, c, d are integers.

This was question B4 in the 56th Putnam International Mathematical Competition in 1995.

The expression within the 8^{th} root is a type of recurring continued fraction with the single partial quotient 2207. Our first step might be to determine its value. Let

$$\psi = a - \cfrac{1}{a - \cfrac{1}{a- \ ...}}$$

so that

$$\psi = a - \frac{1}{\psi} \ , \ \text{ equivalent to } \quad \psi^2 - a\psi + 1 = 0.$$

This quadratic has solutions

$$\psi = \tfrac{1}{2}(a + \sqrt{a^2 - 4}\ ) \ \text{ and } \ \tfrac{1}{\psi} = \tfrac{1}{2}(a - \sqrt{a^2 - 4}\ ) \ .$$

The root factorises and gives the numerical values $\psi = \frac{1}{2}(2207 + \sqrt{2207 \cdot 2209}\ )$.

I propose to find the 8^{th} root of this by taking successive square roots. It seems always worthwhile in these problems to factorise integers so let's note that 2207 is prime, that $2205 = 3^2.5.7^2$ and that $2209 = 47^2$. ψ therefore simplifies to $\frac{1}{2}(2207 + 3.7.47\sqrt{5}\ )$. This lies in the field extension $\mathbb{Q}(\sqrt{5})$ (*i.e.* the rationals $\mathbb{Q}$ augmented with $\sqrt{5}$). It is a property of fields that they are closed under addition, subtraction, multiplication and division, so all powers of ψ will have the form $\alpha + \beta\sqrt{5}$ where α and β are fractions. It seems reasonable to guess that square roots might have the same form, so let us test this by looking for α and β such that

$$(\alpha + \beta\sqrt{5})^2 = \tfrac{1}{2}(2207 + 3.7.47\sqrt{5}\ ) \ .$$

Matching coefficients, we require

$$2(\alpha^2 + 5\beta^2) = 2207 \quad \text{ and } \quad 4\alpha\beta = 3.7.47.$$

To produce odd numbers it is clear that both α and β must be integer multiples of $\frac{1}{2}$. Thus, let $\alpha = A/2$ and $\beta = B/2$. Now

$$A^2 + 5B^2 = 2.2207 \quad \text{ and } \quad AB = 3.7.47.$$

Some easy trials of the possibilities confirm that $A = 47$ and $B = 3.7$. We therefore have determined that

$$\sqrt{\psi} = \tfrac{1}{2}(47 + 21\sqrt{5}\ ) \ .$$

Continuing in the same way we find that

$$\sqrt[4]{\psi} = \tfrac{1}{2}(7 + 3\sqrt{5}\ ) \quad \text{ and } \quad \sqrt[8]{\psi} = \tfrac{1}{2}(3 + \sqrt{5}\ ).$$

So the answer to the Putnam question is that $a = 3$, $b = 1$, $c = 5$ and $d = 2$.

The topic is worth exploring a little further. Taking the next square root gives $\sqrt[16]{\psi} = \frac{1}{2}(1+\sqrt{5}\ )$ which we recognise as the Golden Mean, G, so popular with puzzle-setters. This famous number has the singular continued fraction

$$1 + \cfrac{1}{1 + \frac{1}{1+\ \dots}}$$

which is usually written as $\{1 : \underline{1}\}$ where the underline means 'recurring' . (Note the + signs here compared with the − signs in the question.) Indeed, some insight into the mind of those who compile maths puzzles might have led us to guess that this continued fraction is the one they would be most likely to choose. Suppose, therefore, that we had guessed that the answer involves some power of the Golden Mean – we would then need to construct the given 2207 continued fraction from it.

G satisfies the equation

$$G = 1 + \frac{1}{G} \quad \text{equivalent to} \quad G^2 - G - 1 = 0.$$

Write this as

$$(G^2 - 1) - G = 0 \quad \text{from which} \quad (G^2 - 1) + G = 2G.$$

Multiplying the respective sides of these two equations gives the 'difference of two squares' form

$$(G^2 - 1)^2 - G^2 = 0 \quad \text{which is} \quad (G^2)^2 - 3G^2 + 1 = 0.$$

This is now a quadratic in G^2 and has solution $G^2 = \frac{1}{2}(3+\sqrt{5}\ )$, in agreement with our deduction above. Continuing in the same way,

$$(G^4)^2 - 7\,G^4 + 1 = 0\ , \quad (G^8)^2 - 47\,G^8 + 1 = 0 \quad \text{and} \quad (G^{16})^2 - 2207\,G^{16} + 1 = 0.$$

The pattern of coefficients is that $3^2 - 2 = 7$, $7^2 - 2 = 47$, $47^2 - 2 = 2207$, so the next power will satisfy $(G^{32})^2 - 4870847\,G^{32} + 1 = 0$.

The equation $(G^2)^2 - 3G^2 + 1 = 0$ can be rearranged as $G^2 = 3 - \frac{1}{G^2}$. This would correspond to the continued fraction

$$G^2 = 3 - \cfrac{1}{3 - \frac{1}{3-\ \dots}}\ .$$

Similarly

$$(G^4)^2 - 7\,G^4 + 1 = 0 \quad \text{corresponds to} \quad G^4 = 7 - \cfrac{1}{7 - \frac{1}{7-\ \dots}}\ ,$$

$$(G^8)^2 - 47\,G^8 + 1 = 0 \quad \text{corresponds to} \quad G^8 = 47 - \cfrac{1}{47 - \frac{1}{47-\ \dots}} \quad \text{and finally}$$

$$(G^{16})^2 - 2207\,G^{16} + 1 = 0 \quad \text{corresponds to} \quad G^{16} = 2207 - \cfrac{1}{2207 - \frac{1}{2207-\ \dots}}\ .$$

We have returned to the given problem and confirmed that the solution is G^2.

Here is a final thought. 1/2207 is quite small compared with 2207 so G^{16} must be just less that 2207. In fact the third convergent, equal to

$$2207 - \cfrac{1}{2207 - \frac{1}{2207}} = \frac{10749959329}{4870848} = 2206{\cdot}999546896146\ldots$$

gives the numerical value of G^{16} correct to the 12^{th} decimal place ; the last digit should be 7. It is well recognised that high powers of G are examples of 'almost integers', that is, irrational numbers which are tantalisingly close to whole numbers. The continued fraction representation makes clear why this is so. Higher powers are even closer to being integers : $G^{20} = 15126{\cdot}9999339$ and $G^{21} = 24476 \cdot 00004086$. See the article on Almost Integers in Wikipedia.

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