Q11 : Rationalise the denominator of the fraction

$$\frac{1}{\sqrt[2]{2}+\sqrt[3]{3}+\sqrt[5]{5}.}$$

I found this on a web site. I cannot see a 'clever' way to evaluate this, only a tedious way. Nevertheless, it affords me the opportunity to re-acquaint myself with algebraic extensions to the field of rational numbers, $\mathbb{Q}$. Because the only method I know for rationalising surds is algebraically tedious, I propose only to answer the simpler problem of rationalising

$$\frac{1}{\sqrt[2]{2}+\sqrt[3]{3}.}$$

This denominator lies in the field $\mathbb{Q}(\sqrt{2},\sqrt[3]{3})$ – that is, the rationals augmented by $\sqrt{2}$ and $\sqrt[3]{3}$. All members of this field are a linear combination of $\sqrt{2}$, $\sqrt[3]{3}$ and their powers, meaning that each has the form

$$A+B\sqrt{2}+C\sqrt[3]{3}+D\sqrt[3]{3^2}+E\sqrt{2}\,\sqrt[3]{3}+F\sqrt{2}\,\sqrt[3]{3^2}\ .$$

The rationalised form of our given expression must also be such a linear combination, so we have to find A, B, C, D, E and F such that

$$(\alpha+\beta)(A+B\alpha+C\beta+D\beta^2+E\alpha\beta+F\alpha\beta^2)=1$$

where $\alpha=\sqrt{2}$, $\beta=\sqrt[3]{3}$, $\alpha^2=2$, $\beta^3=3$. From here on it is a tedious process of multiplying this out, equating coefficients of like terms to produce six simultaneous linear equations, and solving the system by, say, a matrix inversion. Here are the various steps :

1) Multiplying the brackets :

$$(A\alpha+B+C\alpha\beta+D\alpha\beta^2+E\beta+F\beta^2)+(A\beta+B\alpha\beta+C\beta^2+D+E\alpha\beta^2+F\alpha)=1$$

2) Equating coefficients :

$$\begin{aligned}
\alpha^0:&\quad 2B+3D &&=1\\
\alpha^1:&\quad A+3F &&=0\\
\beta^1:&\quad 2E+A &&=0\\
\beta^2:&\quad 2F+C &&=0\\
\alpha\beta:&\quad C+B &&=0\\
\alpha\beta^2:&\quad D+E &&=0
\end{aligned}\tag{1}$$

3) Solving simultaneously : I will not give details of this. The result is

$$A=6,\quad B=-4,\quad C=4,\quad ,D=3,\quad E=-3,\quad F=-2.$$

4) The answer :

$$\frac{1}{\alpha+\beta}=6-4\alpha+4\beta+3\beta^2-3\alpha\beta-2\alpha\beta^2$$

as may be confirmed by multiplication.

I don't see enough symmetry in this result to suggest a witty trick for arriving at it. I therefore see little intrinsic interest in this 'puzzle' and don't feel inclined to thrash out the initial problem which will involve $\gamma = \sqrt[5]{5}$ and all combinations with γ^2, γ^3 and γ^4, giving 30 terms in all. What is perhaps of more interest is to determine whether $\mathbb{Q}(\alpha,\beta)$ equals the field $\mathbb{Q}(\alpha+\beta)$, and to find its minimum polynomial.

The extension field $\mathbb{Q}(\alpha+\beta)$ will be identical with $\mathbb{Q}(\alpha,\beta)$ if both α and β can be expressed solely in terms of $\alpha+\beta$, since this will allow β^2, $\alpha\beta$ and $\alpha\beta^2$ and hence all field members also to be expressed. Let $\theta = \alpha+\beta$. We find powers of θ up to θ^6 and search for a linear combination which will equal α and another equal to β. Thus

$$\begin{aligned} \theta^2 &= \alpha^2 + 2\alpha\beta + \beta^2 &&= 2 + \beta^2 + 2\alpha\beta. \\ \theta^3 &= \alpha^3 + 3\alpha^2\beta + 3\alpha\beta^2 + \beta^3 &&= 3 + 2\alpha + 3\alpha\beta^2, \quad \text{etc.} \end{aligned}$$

Continuing, we can collect the coefficients into a matrix equation

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 1 & 2 & 0 \\ 3 & 2 & 6 & 0 & 0 & 3 \\ 4 & 12 & 3 & 12 & 8 & 0 \\ 60 & 4 & 20 & 3 & 15 & 20 \\ 17 & 120 & 90 & 60 & 24 & 18 \end{pmatrix} \begin{pmatrix} 1 \\ \alpha \\ \beta \\ \beta^2 \\ \alpha\beta \\ \alpha\beta^2 \end{pmatrix} = \begin{pmatrix} \theta^1 \\ \theta^2 \\ \theta^3 \\ \theta^4 \\ \theta^5 \\ \theta^6 \end{pmatrix}.$$

Solution of this is best done using algebraic software. One finds the inverse matrix of coefficients and then multiplies on the left by it. The result is

$$\begin{aligned} 36\theta - 12\theta^2 + 6\theta^3 + 6\theta^4 - \theta^6 &= 1 \\ -384433\theta + 126366\theta^2 - 6872\theta^3 - 6525\theta^4 + 48\theta^5 + 1092\theta^6 &= 755\alpha \\ 391888\theta - 126366\theta^2 + 6872\theta^3 + 6525\theta^4 - 48\theta^5 - 1092\theta^6 &= 755\beta. \end{aligned}$$

Clumsy though this appears, it does confirm that $\mathbb{Q}(\alpha+\beta) \equiv \mathbb{Q}(\alpha,\beta)$. Extension of $\mathbb{Q}$ by a single surd is called a 'simple' extension. Moreover, the polynomial $1 - 36\theta + 12\theta^2 - 6\theta^3 - 6\theta^4 + \theta^6$ is irreducible over $\mathbb{Q}$ so must be the minimum polynomial of $\mathbb{Q}(\theta)$; that is, the polynomial of lowest degree with integer coefficients and θ as one root. Indeed, numeric solution confirms that one real root is $\alpha+\beta$ and the other is $-\alpha+\beta$, the other four roots being complex.

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