Q12 : Derive the definite integral

$$\int_0^\infty \frac{\sin^2(a\ln x)}{(x-1)^2}dx = \pi a\coth(2\pi a) - \tfrac{1}{2}.$$

for $a > 0$ real.

This question was posed to me by M. Gilbert de Craemer of Belgium. He had found it stated without proof in 'Recueil de problèmes sur la théorie des fonctions analytiques' which is a French translation of a Russian book of integrals compiled by M. Evgrafov (MIR publishers, Moscow).

Form of the integrand

The integrand is badly behaved near $x = 0$ since here $\ln x \to -\infty$ causing $\sin(\ln x)$ to undergo an infinity of oscillations between $+1$ and -1. This is the only singular behaviour of the integrand since there is no pole at $x = 1$, despite the $x - 1$ denominator. To see this

$$\lim_{\delta\to 0}\frac{\sin^2(a\ln(1+\delta))}{\delta^2} = \lim_{\delta\to 0}\frac{\sin^2(a\delta)}{\delta^2} = \lim_{\delta\to 0}\frac{(a\delta)^2}{\delta^2} = a^2.$$

Change of variable

My colleague Dr. Graham Little pointed out that this singular behaviour near $x = 0$ can be removed by the change of variable $\ln x = 2t$. This gives

$$x = e^{2t}, \qquad dx = 2e^{2t}dt,$$

$$\int_{-\infty}^\infty \frac{\sin^2(2at)}{(e^{2t}-1)^2}\,2e^{2t}\,dt \;=\; 4\int_0^\infty \frac{\sin^2(2at)}{(e^{2t}-1)^2}\,e^{2t}\,dt, \qquad 1)$$

using the symmetry about $x = 0$. We will work with this version of the integral, but it is worth noting that it can be expressed in yet another form by replacing the exponentials by a hyperbolic function :

$$\sinh t = \tfrac{1}{2}(e^t - e^{-t}),$$

$$\frac{e^{2t}}{(e^{2t}-1)^2} = \frac{1}{e^{2t}-2+e^{-2t}} = \frac{1}{(e^t-e^{-t})^2} = \frac{1}{4\sinh t}.$$

Thus the given integral can also be expressed as

$$\int_0^\infty \frac{\sin^2(2at)}{\sinh^2 t}\,dt\ . \qquad 2)$$

Note that $\lim_{t\to 0}$ of the integrand is $4a^2$. Figure 1 is a graph of the integrand in Eqs 1) and 2) for $a = \frac{1}{2}$. It shows a remarkable smooth bell-shaped curve in which the excursions to negative values of y are highly attenuated by the growth of $\sinh^2 t$ with t. It is very similar to the familiar Gaussian normal distribution curve of statistics and indeed is closely approximated, for $a = \frac{1}{2}$, by $\exp(\frac{-2t^2}{3})$.

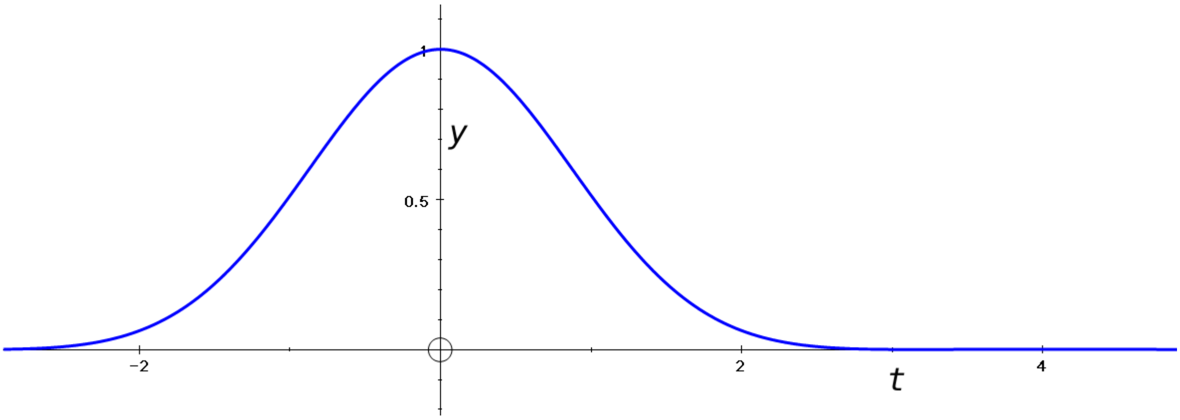

FIGURE 1 – Graph of $y = \frac{\sin^2 t}{\sinh^2 t}$.

Series expansion
Return now to Eq 1). As $t \to \infty$, the denominator tends to e^{4t} and so the asymptotic form of the integrand is $\sin^2(2at)e^{-2t}$. This suggests that we might express the integrand as a power series in e^{-2t} which would give good accuracy not just as $x \to \infty$ but for finite, small values of x, approaching $x = 0$. Therefore, using the binimial theorem which is valid for $\exp(-2t) < 1$,

$$\frac{e^{2t}}{(e^{2t}-1)^2} \;=\; e^{-2t}(1-e^{-2t})^{-2} \;\approx\; e^{-2t}(1+2e^{-2t}+3e^{-4t}+4e^{-6t}+\dots).$$

The integral Eq 1) has now been replaced by

$$4\lim_{\xi\to 0}\int_{\xi}^{\infty}\sum_{k=1}^{\infty}\sin^2(2at)ke^{-2kt}\;dt. \qquad 3)$$

At this stage we must make the assumption that we can interchange the order of integration and summation and then take the limit $\xi \to 0$. This can be justified by showing suitable conditions of convergence of the series and I shall return to this at the end. For the time being, let us press on boldly. Each term in the sum can be integrated by parts or by taking the real part of a complex exponential, as follows :

$$\int \sin^2(2at)e^{-2kt}\;dt = \int \tfrac{1}{2}(1-\cos(4at))e^{-2kt}dt$$

$$= \;-\frac{e^{-2kt}}{4k} - \tfrac{1}{2}\Re\int \exp(-2kt+4iat)dt$$

$$= \;-\frac{e^{-2kt}}{4k} - \tfrac{1}{2}\Re\frac{\exp(-2kt+4iat)}{-2k+4ia}$$

$$= \;-\frac{e^{-2kt}}{4k} + \frac{e^{-2kt}}{2}\Re\frac{(2k+4ia)[\cos(4at)+i\sin(4at)]}{4k^2+16a^2}$$

$$= -\frac{e^{-2kt}}{4k} + \frac{e^{-2kt}}{(4k^2+16a^2)}[k\cos(4at) - 2a\sin(4at)].$$

The limiting value as $t \to \infty$ is 0, and as $t \to 0$ is

$$-\frac{1}{4k} + \frac{k}{(4k^2+16a^2)} = \frac{-a^2}{k(k^2+4a^2)} \ .$$

From this the required integral is

$$\int_0^\infty \frac{\sin^2(a\ln x)}{(x-1)^2}dx \ = \ \int_0^\infty \frac{\sin^2(2at)}{\sinh^2 t}dt \ = \ \sum_{k=1}^{\infty} \frac{4a^2}{k^2+4a^2}. \qquad 4)$$

Evaluation of infinite sum

The next and final step is to evaluate this infinite sum. This can be done by complex integration, using the method described clearly by Dr. John Reade in his book 'Calculus with Complex Numbers', page 71 (pub. Taylor and Francis, 2003). We consider the integral

$$\int_{\gamma(N)} \frac{\cot(\beta z)}{z^2+C^2}\, dz$$

where $\gamma(N) = \gamma_1+\gamma_2+\gamma_3+\gamma_4$ is a square contour in the z plane, centre 0 and half side length $(N+\frac{1}{2})\frac{\pi}{\beta}$, as illustrated in Figure 2.

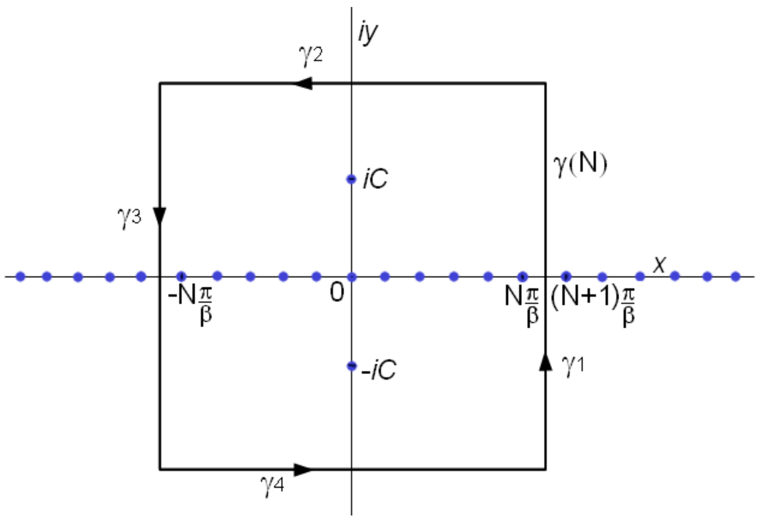

FIGURE 2 – Square contour $\gamma(N)$ enclosing poles of $\frac{\cot(\beta z)}{z^2+C^2}$. Poles are illustrated in blue.

The singularities of $\cot(\beta z)$ are at $\tan(\beta z) = 0$; these are on the real axis at $z = \frac{k\pi}{\beta}$ for k any integer. By drawing the square contour $\gamma(N)$ we enclose the singularity at the origin plus the

N singularities to the left of the origin and the N to the right. By the Residue Theorem, the integral round $\gamma(N)$ must be

$$\oint_{\gamma(N)} = \frac{\cot(\beta z)}{z^2 + C^2}\, dz = 2\pi i \sum_j \mathcal{R}_j \qquad 5)$$

where $\mathcal{R}_j$ are the residues within the contour. Residues arise both from the $\cot \beta z$ and $z^2 + C^2$ factors.

The Laurent expansion of $\cot(\beta z)$ about $z = 0$ is

$$\cot(\beta z) \;\approx\; \frac{1}{\beta z} - \frac{\beta z}{3} + \frac{\beta^3 z^3}{45} - \ldots.$$

The periodic symmetry of $\cot(\beta z)$ means that it has identical behaviour near $z = k\pi/\beta$. Thus near $z = k\pi/\beta$ the integrand behaves as

$$\lim_{\delta \to 0} \frac{1}{\beta\delta[(\frac{k\pi}{\beta} + \delta)^2 + C^2]} \;.$$

Therefore the residue at $z = k\pi/\beta$ is

$$\frac{\beta}{k^2\pi^2 + \beta^2 C^2} \;.$$

Within the square contour there are $2N + 1$ residues with these values.

The poles due to the $z^2 + C^2$ denominator are at $z = \pm iC$. (We assume that $C < N$ so these poles always lie within the square contour.) Near these poles the integrand behaves as

$$\lim_{\delta \to 0} \frac{\cot(\beta(\delta \pm iC))}{(\delta \pm iC)^2 + C^2}$$

so the residues are

$$\pm\frac{\cot(i\beta C)}{\pm 2iC} = -\frac{\coth \beta C}{2C}$$

at both poles.

Putting all these residues into Eq 5) :

$$\oint_{\gamma(N)} \frac{\cot(\beta z)}{z^2 + C^2}\, dz \;=\; 2\pi i \left[\sum_{k=-N}^{N} \frac{\beta}{(k^2\pi^2 + \beta^2 C^2)} - \frac{\coth \beta C}{C} \right]. \qquad 6)$$

Bounds on integral round $\gamma(N)$
Now we consider the limit $N \to \infty$ as the square contour expands to infinity. We shall use the inequality that the integral round $\gamma(N) \leq$ (length of contour)$\times$(maximum absolute value on $\gamma(N)$). Let us deal with the contributions to this inequality in turn. The perimeter of the contour is $8(N+\frac{1}{2})\pi/\beta$. Next note that

$$\frac{1}{|z^2+C^2|} < \frac{1}{|z^2|} < \frac{\beta^2}{N^2\pi^2}.$$

Third, as $N \to \infty$, the limiting maximum value of $|\cot(\beta z)|$ on $\gamma(N)$ is 1. To see this, start from the identity

$$|\cot(\beta z)|^2 = \frac{1+\tan^2(\beta x)\tanh^2(\beta y)}{\tan^2(\beta x)+\tanh^2(\beta y)}.$$

1. On the vertical sides, γ_1 and γ_3, the real part of z is fixed at $x = (N+\frac{1}{2})\pi/\beta$. Here $\tan(\beta x) \to \infty$ so $|\cot(\beta z)| \to \tanh(\beta y)$. This tends to 1 as $y \to \infty$.

2. On the horizontal sides, γ_2 and γ_4, the imaginary part of z is fixed at $y = (N+\frac{1}{2})\pi/\beta$. Here $\tanh(\beta y) \leq \tanh((N+\frac{1}{2})\pi)$ and this tends to 1 as $N \to \infty$. Then $|\cot(\beta z)|^2 \to (1+\tan^2(\beta x))/(\tan^2(\beta x)+1) = 1$.

Collecting these limits and inequalities together, as $N \to \infty$

$$\oint_{\gamma(N)} \frac{\cot(\beta z)}{z^2+C^2}\,dz \;<\; 8(N+\tfrac{1}{2})\frac{\pi}{\beta}\cdot\frac{\beta^2}{N^2\pi^2}\,.\,1 \to 0.$$

Concluding evaluation of the original integral
Now that we have shown that the contour integral is zero, we have from Eq 6)

$$\sum_{k=-\infty}^{\infty} \frac{\beta}{k^2\pi^2+\beta^2C^2} = \frac{\coth\beta C}{C}\,. \qquad 7)$$

This general result can be matched to the sum in Eq 6 by setting $\beta = 2a$, $C = \pi$. This gives

$$\sum_{k=-N}^{N} \frac{2a}{k^2+(2a)^2} = \frac{\pi^2\coth(2\pi a)}{\pi}\,.$$

Single out the term for $k = 0$ and use symmetry about $k = 0$ to obtain, finally,

$$1+2\sum_{k=1}^{N}\frac{2a}{k^2+4a^2} = \pi\coth(2\pi a),$$

$$\int_0^\infty \frac{\sin^2(a\ln x)}{(x-1)^2}dx \;=\; \int_0^\infty \frac{\sin^2(2at)}{\sinh^2 t}dt \;=\; \sum_{k=1}^\infty \frac{4a^2}{k^2+4a^2} = \pi a\coth(2\pi a) - \tfrac{1}{2}.$$

Convergence of series in Eq 3)

In Eq 3) we are seeking a three-stage limiting process : i) to commute the order of summation and integration, and carry out the integration first, then ii) sum to infinity over k, and iii) take the lower limit ξ to zero. I shall examine two types of convergence of this series over the semi-open interval $[0, \infty)$, :

1. point-wise convergence at each fixed value of t,
2. dominated convergence according to Lebesgue's theorem.

Dominated convergence, which first requires point-wise convergence, is sufficient to justify interchangeability of summation and integration in the Lebesgue sense, and hence to justify the operations on Eq 3). Note that all terms are positive definite so convergence also means absolute convergence. Note also that at $t = 0$ the value is clearly 0 so the series converges in all senses at $t = 0$.

Point-wise convergence : Let the term in Eq 3) be $f(t,k) = \sin^2(2at)ke^{-2kt}$. Fix t. For $t = t_0 > 0$

$$\sum_{k=1}^\infty f(t_0,k) = \sin^2(2at_0)\sum_{k=1}^\infty ke^{-2kt_0} \;.$$

All terms have the same sign and the ratio of adjacent terms is

$$\frac{(k+1)e^{-2(k+1)t_0}}{ke^{-2kt_0}} = \frac{k+1}{k}e^{-2k} \to 0$$

as $k \to \infty$. Thus the sequence of terms converges absolutely to zero by the Ratio Test. Now consider a sequence derived from this – the sequence of partial sums $p(t,K) = \sum_{k=1}^K f(t,k)$. The remainder after K terms is $\sum_{k=K+1}^\infty f(t,k)$. We can show that this remainder can always be made as small as we choose by taking K large enough :

$$k < e^{kt} \;\text{ so }\; ke^{-2kt} < e^{-kt} \;\text{ and }\; \sum_{k=K+1}^\infty f(t,k) < \sum_{k=K+1}^\infty e^{-kt}.$$

This is a geometric series, first term $e^{-(K+1)t}$, common ratio e^{-kt}, and hence sum

$$\frac{e^{-(K+1)t}}{1-e^{-t}} \to 0 \;\text{ as }\; K \to \infty.$$

Specifically, the remainder is $< \epsilon$ (> 0) if

$$K > \frac{1}{t}\ln\frac{1}{\epsilon(1-e^{-t})}.$$

Therefore the series in Eq 3) converges pointwise over $[0, \infty)$.

Dominated convergence : Confident that the series of partial sums $p(t,k)$ over k converges point-wise, we now look for an integrable function $g(t)$ over $(0,\infty)$ which 'dominates' $p(t,K)$ for all K and t. 'Dominates' means that $p(t,K) \leq g(t)$ for all $K \geq 1$ and all $t > 0$. One approach to finding a suitable $g(t)$ is as follows :

$$\sum_{k=1}^{K} \sin^2(2at) k e^{-2kt} < \int_0^\infty k e^{-2kt} dk = \frac{(2kt+1)e^{-2kt}}{4t^2} |_0^\infty = \frac{1}{4t^2}.$$

So we choose $g(t) = 1/(4t^2)$. This is integrable for all $t > 0$. Hence, by the dominated convergence theorem

$$\lim_{K\to\infty} \int_\xi^\infty p(t,K)dt = \int_\xi^\infty F(t)dt$$

where $F(t)$ is the function defined by the limit of the sequence of partial sums. But, of course, this is our required function from Eqs 2) and 3), namely

$$\frac{1}{4}\frac{\sin^2(2at)}{\sinh^2 t} \ .$$

This justifies the procedure I have used and, as a bonus, shows that $\sin^2(2at)/\sinh^2 t$ is bounded above by $1/t^2$.

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