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Q13 : Evaluate

$$I = \int_2^4 \frac{\sqrt{\ln(9-x)}}{\sqrt{\ln(9-x)} + \sqrt{\ln(x+3)}}\, dx$$

This was problem B-1 in the 1987 William Lowell Putnam mathematical competition. There is interest and amusement value in these problems only if the answer is some common value such as 0, 1 , π, e, *etc*, so we expect somthing like this here. Moreover, Putnam puzzles can often be solved by some 'trick' say by appeal to symmetry, or to the pigeon hole principle, or similar overarching principle.

Here the function $\sqrt{\ln(9-x)}$ looks so improbable that we might suspect that it is not material to the solution. Furthermore, when $x = 3$, both $9 - x$ and $x + 3$ have the value 6, whilst $x = 3$ is the mean of the two limits of integration. This, therefore, looks like symmetry about $x = 3$. Accordingly, let's make a) the transformation to $u = x - 3$, and b) generalise $\sqrt{\ln(9-x)}$ to $f(x)$.

$$I = \int_{-1}^{1} \frac{\sqrt{\ln(6-u)}}{\sqrt{\ln(6-u)} + \sqrt{\ln(6+u)}}\, du = \int_{-1}^{1} \frac{f(6-u)}{f(6-u) + f(6+u)}\, du. \qquad 1)$$

The numerator is clearly a term in the denominator, and the denominator is symmetric about $u = 0$. Hence

$$\int_{-1}^{1} \frac{f(6-u)}{f(6-u) + f(6+u)}\, du \;+\; \int_{-1}^{1} \frac{f(6+u)}{f(6-u) + f(6+u)}\, du = \int_{-1}^{1} 1 du = 2. \qquad 2)$$

Looking further at the symmetry about $u = 0$, consider the effect of substituting $v = -u$:

$$\int_{-1}^{1} \frac{f(6-u)}{f(6-u) + f(6+u)}\, du = \int_{1}^{-1} \frac{f(6+v)}{f(6+v) + f(6-v)}\, (-dv). \qquad 3)$$

But, on swapping the limits of integration and writing u for v, this become precisely the second term on the left of Eq 2). Hence the two terms on the left of Eq 2) are identical, so each must equal 1. We conclude that $I = 1$.

To emphasise that the function $f(x)$ is immaterial to the value of I, I have carried out numerical integration on both $f(t) = \sqrt{\ln(t)}$ and on $f(t) = \sin(t^2)e^t$ and in both cases the quadrature gives $1 \cdot 000$.

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