

Q14 : Evaluate $\mathcal{L} = \int_0^\infty x\mathcal{F}\mathcal{G} dx$ where

$$\mathcal{F} = 1 - \frac{x^2}{2} + \frac{x^4}{2 \cdot 4} - \frac{x^6}{2 \cdot 4 \cdot 6} + \dots \quad \text{and} \quad \mathcal{G} = 1 + \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} + \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \dots$$

This was problem A-3 in the 1997 William Lowell Putnam mathematical competition. We are given an alternating series \mathcal{F} and a positive-definite series \mathcal{G} , and are asked to determine the semi-infinite integral of x times their product. Questions in the Putnam competitions do not arise in ‘real life’ but rather are invented. There is only interest or amusement value in them if the answer is some simple, common value such as 0, 1, π , e , *etc*, so we expect this here.

Multiplying the two infinite series would be very complicated (see §3). I am therefore inclined to see if we can identify \mathcal{F} or \mathcal{G} as a representation of a known continuous function, which can then be integrated by established means.

1 Treatment as series and one known function

In fact \mathcal{F} is quite readily recognised as the MacLaurin series for $\exp(-x^2/2)$. You can see this either from the series for e^u with $u = -x^2/2$, or by noting that \mathcal{F} satisfies the differential equation

$$\frac{d\mathcal{F}}{dx} = -x\mathcal{F} dx \quad \text{with solution} \quad \ln \mathcal{F} = -\frac{x^2}{2} + \text{constant.}$$

The series for \mathcal{G} is not so obvious (see §2) so, as a half-way house, we can explore the functional series formed by multiplying the \mathcal{G} series by $x \exp(-x^2/2)$:

$$\mathcal{L} = \int_0^\infty \left(x e^{-x^2/2} + \frac{x^3}{2^2} e^{-x^2/2} + \frac{x^5}{2^2 \cdot 4^2} e^{-x^2/2} + \frac{x^7}{2^2 \cdot 4^2 \cdot 6^2} e^{-x^2/2} + \dots \right) dx \quad 1)$$

Fortunately, these integrals can be evaluated in closed form. (The corresponding integral with even powers of x cannot – they all involve the error function.) By considering the derivative of $\exp(-c^2x^2)$,

$$\int x e^{-c^2x^2} dx = -\frac{1}{2c^2} e^{-c^2x^2},$$

Now differentiation of both sides of the equation with respect to c gives

$$\int x^3 e^{-c^2x^2} dx = -\frac{(c^2x^2 + 1)}{2c^4} e^{-c^2x^2}.$$

Further differentiation with respect to c gives in sequence the integrals with higher odd powers of x . For instance¹

$$\int x^5 e^{-c^2x^2} dx = -\frac{(c^4x^4 + 2c^2x^2 + 2)}{2c^6} e^{-c^2x^2}.$$

In this way we find that

$$\int x e^{-x^2/2} dx = -e^{-x^2/2}$$

1. Notice how the constant in the polynomial multiplying $e^{-c^2x^2}$ develops through this sequence of integrals. In generating the integral for x^7 the ‘2’ here (for x^5) will be multiplied by the derivative of the denominator, c^6 , so giving a term $2.6c^5$, which is then divided by the $2c$ from the left side of the equation. Moreover, the denominators increase by c^2 through the sequence of integrals, so that for the x^9 integral, the constant 6 will be multiplied by $8c^7$. This is the pattern generating the constant terms, which are all that remain at the limit $x = 0$.

$$\int x^3 e^{-x^2/2} dx = -(x^2 + 2)e^{-x^2/2}$$

$$\int x^5 e^{-x^2/2} dx = -(x^4 + 4x^2 + 8)e^{-x^2/2}$$

$$\int x^7 e^{-x^2/2} dx = -(x^6 + 6x^4 + 24x^2 + 48)e^{-x^2/2}$$

$$\int x^9 e^{-x^2/2} dx = -(x^8 + 8x^6 + 48x^4 + 192x^2 + 384)e^{-x^2/2}$$

Now include the limits of integration. The value of each of these integrals as $x \rightarrow \infty$ is 0. The values at $x = 0$ are given by the constant terms in the polynomials and are respectively 1, 2, 8 = 2.4, 48 = 2.4.6, 384 = 2.4.6.8, *etc.* Putting these into Eq 1,

$$\mathcal{L} = \int_0^\infty x \mathcal{F} \mathcal{G} dx = 1 + \frac{2}{2^2} + \frac{2.4}{2^2 \cdot 4^2} + \frac{2.4.6}{2^2 \cdot 4^2 \cdot 6^2} + \frac{2.4.6.8}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8^2} + \dots$$

which evaluates to

$$1 + \frac{1}{2} + \frac{1}{2.4} + \frac{1}{2.4.6} + \frac{1}{2.4.6.8} + \dots = \sum_{n=0}^{\infty} \frac{1}{2^n n!}.$$

Comparing this with the series for e^t , we can identify \mathcal{L} with $e^{\frac{1}{2}} = \sqrt{e}$, with numerical value 1.648721271 .

2 Treatment as two known functions

The series \mathcal{G} is the MacLaurin expansion for $I_0(x)$, the modified Bessel function of the first kind. The clue to spotting this is the factorial squared in the denominator – Bessel functions are the most common functions with a product of factorials in the denominators of their power series expansions. The non-oscillatory modified Bessel function $I_0(x)$ is related to the oscillatory $J_0(x)$ much as $\cosh(x)$ is related to $\cos(x)$, namely $I_0(x) = J_0(ix)$. In formula 11.4.29 on page 486 of ‘Handbook of Mathematical Functions’, Abramowitz and Stegun quote the standard integral²

$$\int_0^\infty e^{-at^2} t^{\nu+1} J_\nu(bt) dt = \frac{b^\nu}{(2a)^{\nu+1}} e^{-\frac{b^2}{4a}}, \quad \Re \nu > -1, \quad \Re a > 0. \quad 2)$$

Make the substitutions $\nu = 0$, $a = 1/2$, $b = i$ and we get

$$\int_0^\infty e^{-t^2/2} t I_0(t) dt = e^{\frac{1}{2}},$$

which is the result obtained in §1. Of course, this all begs the question of how Eq 2 has been derived.

Related formulae are to be found in the encyclopaedic ‘Table of integrals, series and products’ by Gradshteyn and Ryzhik, edited by Jeffrey, Academic Press, 1965. On page 717, entries 6.631.7, 8 and 9 read

$$\int_0^\infty x e^{-ax^2} J_\nu(bx) dx = \frac{b\sqrt{\pi}}{8a^{3/2}} \exp\left(-\frac{b^2}{8a}\right) \left[I_{\frac{\nu-1}{2}}\left(\frac{b^2}{8a}\right) - I_{\frac{\nu+1}{2}}\left(\frac{b^2}{8a}\right) \right] \quad \Re a > 0, \quad \Re \nu > -2. \quad 3)$$

2. I have replaced their a^2 by a to simplify the notation.

$$\int_0^1 x^{n+1} e^{-ax^2} I_n(2ax) dx = \frac{1}{4a} \left[e^a - e^{-a} \sum_{r=-n}^n I_r(2a) \right], \quad n = 0, 1, 2, \dots \quad 4a)$$

$$\int_1^\infty x^{1-n} e^{-ax^2} I_n(2ax) dx = \frac{1}{4a} \left[e^a - e^{-a} \sum_{r=1-n}^{n-1} I_r(2a) \right], \quad n = 1, 2, \dots \quad 4b)$$

Let's check the value obtained from Eq 3 with $\nu = 0$, $a = 1/2$, $b = i$.

$$\int_0^\infty x e^{-x^2/2} I_0(x) dx = i 2\sqrt{2\pi} \exp(\frac{1}{4}) \left[I_{-\frac{1}{2}}(\frac{1}{4}) - I_{\frac{1}{2}}(\frac{1}{4}) \right].$$

We now need some properties of Bessel functions of order $\pm\frac{1}{2}$. Abramowitz and Stegun give³

$$I_{-\nu}(z) - I_\nu(z) = \frac{2 \sin(\nu\pi)}{\pi} K_\nu(z), \quad \text{page 375, 9.6.2}$$

$$\text{where } K_{\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2z}} e^{-z}, \quad \text{page 444, 10.2.17}$$

These give

$$I_{-\frac{1}{2}}(z) - I_{\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} e^{-z}.$$

$$\text{Finally } i \sqrt{\frac{2}{\pi}} \exp(\frac{1}{4}) \sqrt{\frac{2}{\pi(-\frac{1}{4})}} \exp(\frac{1}{4}) = e^{\frac{1}{2}} \quad \text{as before.}$$

The derivation of Eqs 2, 3 and 4a, b must start with a definition of the Bessel function, and that takes us back to an infinite series. Here is a proof of an integral derived Eq 2. Use $J_\nu(iz) = \exp(\nu\pi i/2) I_\nu(z) = i^\nu I_\nu(z)$ (Abramowitz and Stegun 9.6.3, page 375) to obtain

$$\int_0^\infty e^{-at^2} t^{\nu+1} I_\nu(bt) dt = \frac{b^\nu}{(2a)^{\nu+1}} e^{\frac{b^2}{4a}}. \quad \Re\nu > -1, \quad \Re a > 0. \quad 5)$$

$I_\nu(z)$ is defined by the infinite series

$$I_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{2k}}{k! \Gamma(k + \nu + 1)}$$

where the Gamma function is related to the factorial by $k! = \Gamma(k + 1)$, and itself has the integral representation

$$\Gamma(z) = a^z \int_0^\infty s^{z-1} e^{-as} ds, \quad \Re z > 0, \quad \Re a > 0.$$

In the left side of Eq 5, change the order of summation and integration (assumed valid!)

$$\left(\frac{b}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{1}{k! \Gamma(k + \nu + 1)} \int_0^\infty e^{-at^2} t^{\nu+1} t^\nu \left(\frac{bt}{2}\right)^{2k} dt.$$

The integral is

$$\left(\frac{b}{2}\right)^{2k} \int_0^\infty e^{-at^2} t^{2\nu} t^{2k} t dt.$$

3. The same result can be obtained using relations 10.2.13 and 10.2.14 on page 443 of Abramowitz and Stegun.

Now change the variable from t to $s = t^2$ and the integral becomes

$$\left(\frac{b}{2}\right)^{2k} \int_0^\infty e^{-as} s^{k+\nu} t \frac{ds}{2t} = \frac{1}{2} \left(\frac{b}{2}\right)^{2k} a^{-(k+\nu+1)} \Gamma(k+\nu+1).$$

The factors $\Gamma(k+\nu+1)$ cancel and the sum reduces to the MacLaurin series for $\exp(\frac{b^2}{4a})$, then simplifies to the right hand side of Eq 5.

Now that we know that the given integral \mathcal{L} involves a modified Bessel function, we can propose a companion problem based on the ordinary Bessel function of the first kind, $J_0(x)$, the series representation of which is the alternating equivalent of series \mathcal{G} . Hence our new question is

Evaluate $\mathcal{L}' = \int_0^\infty x \mathcal{F} \mathcal{G}' dx$ where

$$\mathcal{F} = 1 - \frac{x^2}{2} + \frac{x^4}{2 \cdot 4} - \frac{x^6}{2 \cdot 4 \cdot 6} + \dots \quad \text{and} \quad \mathcal{G}' = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \dots$$

From Eq 2 the answer is $e^{-1/2} = 0.60653$.

3 Treatment as two series

For completeness, let us see how far we can get by bluntly multiplying out the two given infinite series.

$$\mathcal{F} = 1 - \frac{x^2}{2^1 \cdot 1!} + \frac{x^4}{2^2 \cdot 2!} - \frac{x^6}{2^3 \cdot 3!} + \dots = \sum_{h=0}^{\infty} (-1)^h \frac{x^{2h}}{2^h h!},$$

$$\mathcal{G} = 1 + \frac{x^2}{2^2 \cdot (1!)^2} + \frac{x^4}{2^4 \cdot (2!)^2} + \frac{x^6}{2^6 \cdot (3!)^2} + \dots = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2^k k!)^2}.$$

The structure of the product of these series can be made more clear if we write

$$\beta_k = 2^k k! \quad \text{so that} \quad \mathcal{F} = \sum_{h=0}^{\infty} (-1)^h \frac{x^{2h}}{\beta_h} \quad \text{and} \quad \mathcal{G} = \sum_{k=0}^{\infty} \frac{x^{2k}}{\beta_k^2}.$$

When \mathcal{F} and \mathcal{G} are multiplied, the coefficient of x^{2n} is the sum of all pair-wise combinations of x^{2h} from \mathcal{H} and x^{2k} from \mathcal{G} for which $h+k=n$. Thus in $\mathcal{F}\mathcal{G}$ the coefficient of x^{2n} is

$$(-1)^n \left(\frac{1}{\beta_0^2 \beta_n} - \frac{1}{\beta_1^2 \beta_{n-1}} + \frac{1}{\beta_2^2 \beta_{n-2}} - \frac{1}{\beta_3^2 \beta_{n-3}} + \dots + \frac{1}{\beta_n^2 \beta_0} \right),$$

where $\beta_0 = 1$. The r^{th} term in this bracket ($r = 0, 1, 2, \dots, n$) is

$$\left(\frac{1}{\beta_n} \right) \frac{\beta_n}{\beta_r^2 \beta_{n-r}} = \left(\frac{1}{\beta_n} \right) \frac{2^n}{\beta_r} \frac{n!}{2^r 2^{n-r} r!(n-r)!} = \frac{{}^n C_r}{\beta_n \beta_r}$$

where ${}^n C_r$ are the binomial coefficients 'from n choose r '. We can take $1/\beta_n$ as a factor from each term; however, if we want to make explicit the integer numerators and denominators, we should instead take $1/\beta_n^2$ as a factor. Then the coefficient of x^{2n} is

$$(-1)^n \frac{\sum_{r=0}^n (-1)^r \frac{\beta_n}{\beta_r} {}^n C_r}{\beta_n^2}$$

The integer denominator is β_n^2 and the integer numerator is the sum over r . The first few terms of the product are

$$\mathcal{FG} = 1 - \frac{x^2}{2^2} + \frac{x^4}{8^2} + \frac{7x^6}{48^2} - \frac{127x^8}{384^2} + \frac{1711x^{10}}{3840^2} - \frac{23231x^{12}}{46080^2} + \dots$$

Pressing on, the next step is to integrate. The resulting indefinite integral is

$$\int x\mathcal{FG} dx = \frac{x^2}{2} - \frac{x^4}{4 \cdot 2^2} + \frac{x^6}{6 \cdot 8^2} + \frac{7x^8}{8 \cdot 48^2} - \frac{127x^{10}}{10 \cdot 384^2} + \frac{1711x^{12}}{12 \cdot 3840^2} - \frac{23231x^{14}}{14 \cdot 46080^2} + \dots \quad (6)$$

The n^{th} term ($n = 0, 1, 2, \dots, n$) is

$$\frac{(-1)^n x^{2n+2} \Sigma_n}{(2n+2)\beta_n} \quad \text{where} \quad \Sigma_n = \sum_{r=0}^n (-1)^r \frac{{}^n C_r}{\beta_r} = \sum_{r=0}^n (-1)^r \frac{{}^n C_r}{2^r r!} \quad (7)$$

The function $\Sigma_n(n)$ is interesting in itself and is examined below and in the Appendix.

At the lower limit, $x = 0$, the value is obviously zero. The upper limit must be regarded as the limit $x \rightarrow \infty$ of this series. Before considering how this might be done, we can evaluate the series with $x = 1$ to find the integral from 0 to 1. The first few terms converge rapidly to 0.44040675 which, from Eq 4a, is the value of

$$\int_0^1 x e^{-x^2/2} I_0(x) dx = \frac{1}{2} \left[e^{1/2} - e^{-1/2} I_0(1) \right].$$

So how might we go about taking the limit $x \rightarrow \infty$ of Eq 6, either analytically or numerically? Let's look first at convergence. If the terms in Eq 6 are u_0, u_1 , etc., the ratio of adjacent terms is easily shown to be

$$\left| \frac{u_{n+1}}{u_n} \right| = \frac{x^2}{2n+4} \left| \frac{\Sigma_{n+1}}{\Sigma_n} \right|$$

where Σ_n is defined in Eq 7. Figure 1 is a graph of Σ_n against n for n from 0 to about 300.

The Appendix gives some further analysis of this interesting function. Here we need to note that for many consecutive values of n , Σ_n and Σ_{n+1} have about the same value, or are within about 20% of the same value. This is because the gradient of the curve is nowhere very high. This means that we can take $\Sigma_{n+1}/\Sigma_n \approx 1$. Then

$$\left| \frac{u_{n+1}}{u_n} \right| \approx \frac{x^2}{2n+4}.$$

Absolute convergence of the series of Eq 6 can occur only if this ratio is < 1 , so this identifies the n^{th} term beyond which higher terms decrease and hence from where the series converges. As a practical guide to evaluation, we need to sum at least to the N^{th} term where $N = x^2$ before numerical stability can be expected.

In addition to requiring convergence, we want adequate precision of any evaluation, which means not only that the value of the last term added be sufficiently small, but also that rounding errors are controlled. In fact, unless high precision calculation is used, there is a severe problem with rounding errors in any attempt to sum the series numerically for values of x above

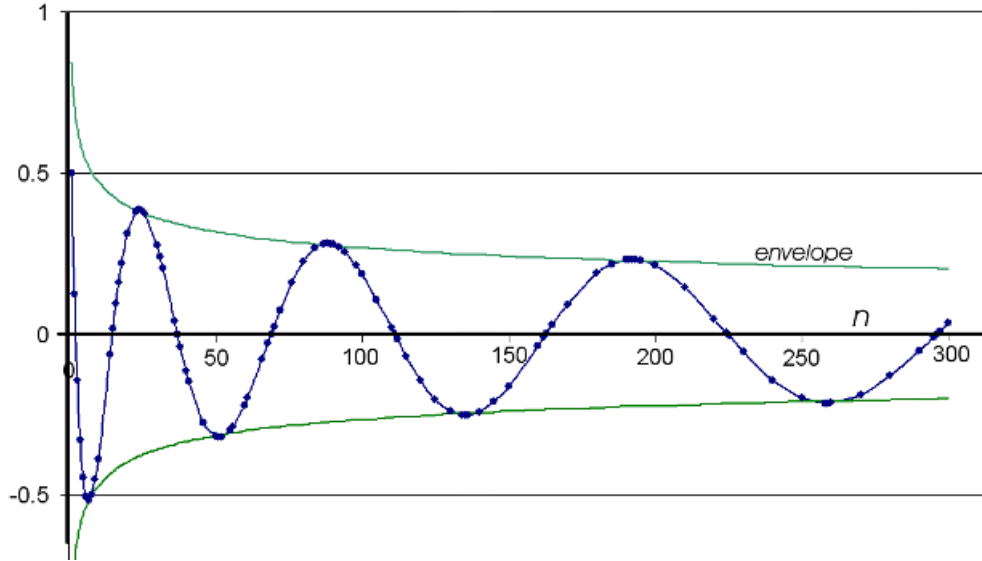


FIGURE 1 – Numerically calculated values of Σ_n against n , with the putative envelopes $1/(2n)^{1/4}$ shown as full lines.

X	Integral	Number of terms
0	0	
0.25	0.031007	4
0.5	0.121136	4
$\frac{1}{\sqrt{2}}$	0.234722	5
1	0.440407	6
$\sqrt{2}$	0.774684	9
1.5	0.844079	9
$\frac{\pi}{2}$	0.900323	10
2	1.205195	13
2.5	1.449467	17
e	1.517367	17
3	1.576646	20
π	1.596833	24
3.5	1.627914	25
4	1.643957	30
5	1.648599	42
6	1.648720	58
2π	1.648721	62
∞	1.648721	

TABLE 1 – Integral $\int_0^X x\mathcal{F}\mathcal{G} dx$ as a function of its upper limit, X .

about 3. This is because the early terms are large and vary in sign – a result is obtained only by massive cancellation of terms. Not surprisingly, I find myself unable to make any progress in summing Eq 6 analytically. The above discussion is a guide to numerical evaluation and accordingly I have used the series Eq 6 to calculate the integral from 0 to an upper limit X for the values of X in Table 1. The last column records the total number of terms required to

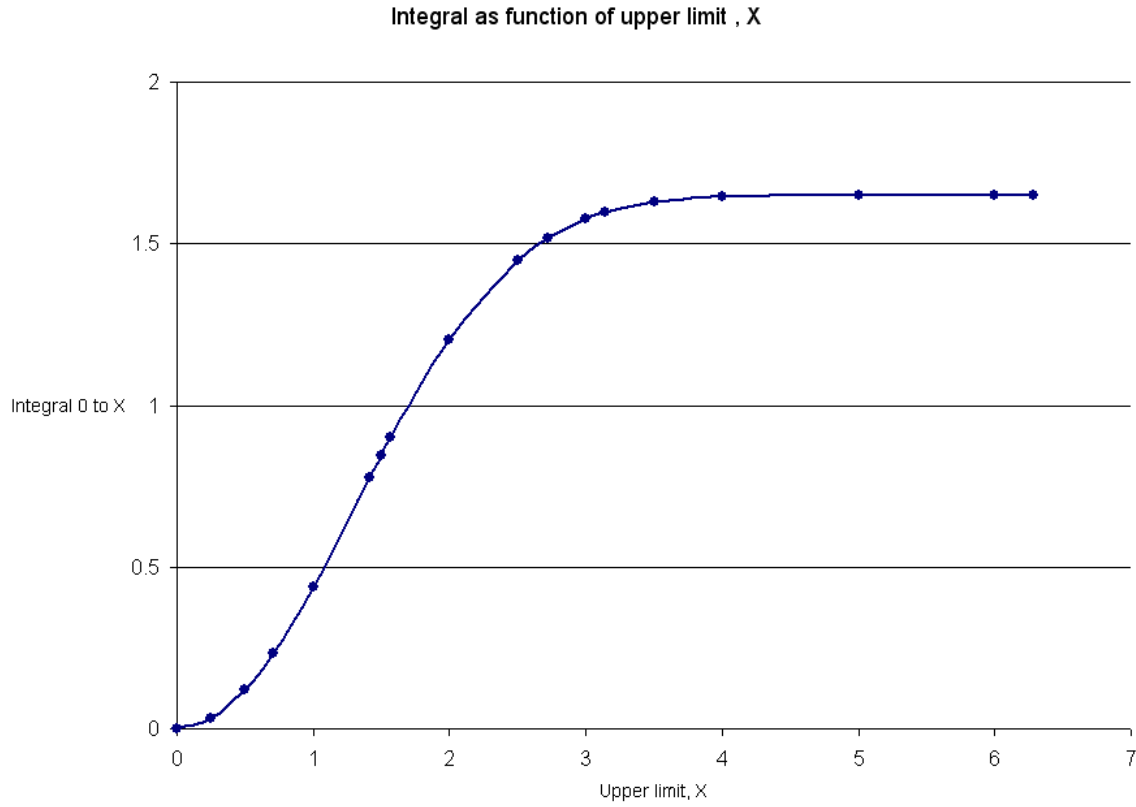


FIGURE 2 – Values of the integral $\int_0^X e^{-t^2/2} t I_0(t) dt$ obtained by summing the series in Eq 6.

gain stability in the sixth decimal place. We already know that as $X \rightarrow \infty$ the integral tends to $\sqrt{e} = 1.6487212707$. Figure 2 is a plot of the values in the table, showing the sigmoidal shape.

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4 Appendix : some analysis of $\Sigma_n(n)$

In this Appendix we are focusing on Σ_n , defined in Eq 7 by a series of $n + 1$ terms. (This is not to be confused with the infinite series for the integral in Eq 6.)

The intriguing feature of this function is its oscillatory graph, with attenuating amplitude and increasing period, plotted in Figure 1. The curve explains why some pairs of adjacent Σ_n , Σ_{n+1} have the same sign, contrary to the general trend of alternating + and -. For example, Σ_2 and Σ_3 , and also Σ_{36} and Σ_{37} are both positive, whilst Σ_{14} and Σ_{15} are both negative. In each case the two values straddle a zero of the curve.

The first few terms in the series for Σ_n are

$$1, \quad -\frac{1}{2}, \quad \frac{1}{16}n(n-1), \quad -\frac{1}{288}n(n-1)(n-2), \quad \frac{1}{9216}n(n-1)(n-2)(n-3), \quad \dots$$

When these are added, they represent Σ_n as a polynomial in n of degree n : $\Sigma_n = \sum_k c_k n^k$. For example, the polynomial for $n = 4$ is

$$1 - \frac{2627}{4608}n + \frac{683}{9216}n^2 - \frac{19}{4608}n^3 + \frac{1}{9216}n^4, \quad 8)$$

$$\text{or in decimal } 1 - 0.570095n + 0.074110n^2 - 0.004123n^3 + 0.000109n^4,$$

which, with $n = 4$, evaluates to -0.330729167 . In principle, for any n , one could sum all terms over r , but numerically only about the first third of terms contribute. This is because the factor $2^n r!$ in the denominator forces a very rapid decrease. I find that the largest term (in absolute value), and the index r of the highest term to exceed 10^{-5} , are as given the Table 2.

TABLE 2 – Σ_n : r values of the largest term in Eq 7, and of the highest term to exceed 10^{-5} .

n	r of largest term	r for which $> 10^{-5}$
10	2	7
20	2, 3	10
30	3	12
40	4	14
50	4	15
60	5	17
70	5	18
80	6	19
100	6	21
200	9	29
300	12	35

Because of the truncated factorial factor $n!/(n-r)!$ within the binomial coefficients ${}^n C_r$ (see Eq 7), the coefficients c_k within such polynomials for Σ_n always work out such that substituting a numerical value n' for n will evaluate to exactly the same value of $\Sigma_{n'}$ whenever $n' \leq n$. For example, the polynomial for Σ_{20} begins

$$1 - 0.570151n + 0.074228n^2 - 0.004207n^3 + 0.000133n^4 - \dots \quad 13 \text{ terms.}$$

and substituting $n = 4$ gives precisely the same value of -0.330729167 as does Eq 8⁴. For this reason, a given polynomial representation of Σ_n for n large is sufficient to ensure exact

4. Note also that the early coefficients, c_k in the polynomial Σ_n , do not change very much with increasing n .

evaluation of $\Sigma_{n'}$ for all $n' \leq n$. Also, we have already noted that Table 2 gives assurance that the polynomial for n will allow high precision evaluation for an n' several times greater than n . Accordingly, Table 3 lists the coefficients of Σ_{62} , these being sufficient to calculate all the integrals listed in Table 1. The polynomial with these coefficients, evaluated over the real numbers, gives the smooth curve in Figure 1.

TABLE 3 – Coefficients c_k in the polynomial expansion of $\Sigma_n(n) = \sum_{k=0}^n c_k n^k$ for $n = 62$.

k	c_k	k	c_k	k	c_k
0	1	21	-2.321251E-46	42	1.471094E-115
1	-0.570151421	22	2.399077E-49	43	-3.978562E-119
2	0.074227584	23	-2.268504E-52	44	1.027640E-122
3	-0.004206533	24	1.969943E-55	45	-2.537665E-126
4	0.000133014	25	-1.576512E-58	46	5.997010E-130
5	-2.68114E-06	26	1.166442E-61	47	-1.357542E-133
6	3.74455E-08	27	-8.002719E-65	48	2.946342E-137
7	-3.83688E-10	28	5.105221E-68	49	-6.136231E-141
8	3.00725E-12	29	-3.036011E-71	50	1.227357E-144
9	-1.86112E-14	30	1.687090E-74	51	-2.359598E-148
10	9.32519E-17	31	-8.779816E-78	52	4.363526E-152
11	-3.86014E-19	32	4.287953E-81	53	-7.767668E-156
12	1.34231E-21	33	-1.969160E-84	54	1.332008E-159
13	-3.97639E-24	34	8.518774E-88	55	-2.201830E-163
14	1.0155E-26	35	-3.477685E-91	56	3.510823E-167
15	-2.25884E-29	36	1.341931E-94	57	-5.403302E-171
16	4.41553E-32	37	-4.901936E-98	58	8.031553E-175
17	-7.64509E-35	38	1.697610E-101	59	-1.153555E-178
18	1.18059E-37	39	-5.581392E-105	60	1.599052E-182
19	-1.63616E-40	40	1.744429E-108	61	-2.097334E-186
20	2.04632E-43	41	-5.189358E-112	62	2.189513E-190

Using the polynomial for $n = 62$ (Table 3) I have explored the maxima, minima and zeros of Σ_n over the range of Figure 1, and these are listed in Table 4.

TABLE 4 – Zeros of Σ_n , plus values of maxima and minima, calculated from Σ_{62} .

Zeros of Σ_{62}	n at Max/Min	Value
2.418811084	6.871273377	-0.518628
14.77457438	24.14751365	0.385683
36.98406453	51.28978785	-0.320756
69.06121393	88.30111438	0.280445
111.0074476	135.1818817	-0.252305
162.8230900	191.9321876	0.231231
224.5082472	258.5520661	-0.214686
296.0629621	335.0415327	0.201251
377.4872547		

From these critical points I have looked to see whether $\Sigma_n(n)$ can be closely represented by an elementary analytic function, rather than the high degree polynomial of Table 3. To my surprise, I find that the zeros occur quite closely as $\sqrt{2n}$, and that absolute values of

maxima and minima fall closely on the graph of $1/\sqrt[4]{2n}$. Incorporating a phase shift to bring the approximating function into alignment,

$$\Sigma_n \approx \frac{1}{(2n)^{1/4}} \cos\left(\sqrt{2n} - \frac{3}{4}\right).$$

Except for the first few terms, the agreement is close; the error is typically less than 3% for most data points in Figure 1. Closer approximations can of course be obtained at the expense of greater complexity; for example

$$\Sigma_n \approx \frac{1}{(2n)^{1/4}} \cos\left(\sqrt{2n + \frac{1}{\sqrt{2}}} - \frac{\pi}{4}\right).$$

Finally, we can look further at the values of the coefficients c_k in the polynomial $\Sigma_n(n) = \sum_{k=0}^n c_k n^k$. The coefficients for $n = 62$ are close to those for other values, n' , except for the last few c_k close to $k = n'$. The logarithms of absolute values $n = 62$ are plotted in Figure 3. They form a graceful curve which tends roughly towards a straight line of gradient about 9. Over the range $k = 30$ to 60 $\ln |c_k|$ can be modelled roughly by

$$\ln |c_k| \approx \frac{16607 - 2586k}{k - 390}$$

though this is merely a roughly fitted curve with no further significance.

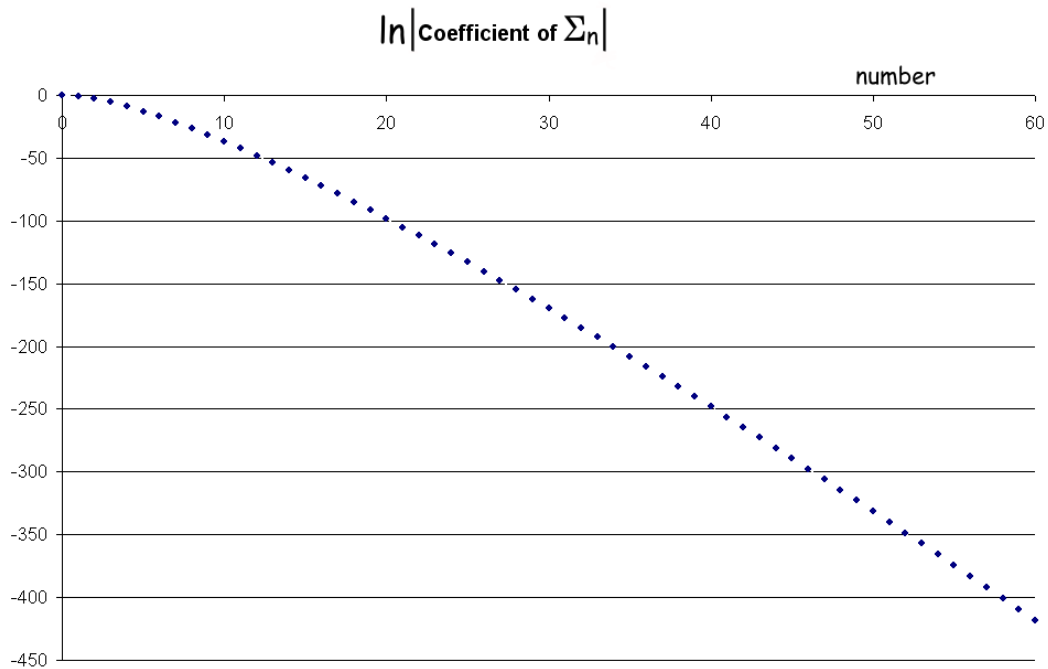


FIGURE 3 – Logarithm of $|c_k|$.