Q17 : Let N be the positive integer with 1998 decimal digits, all of them 1. That is N= 111111 .... 111. Find the thousandth digit after the decimal point of $\sqrt{N}$.

This was problem B-5 in the 1998 Putnam mathematical competition – hence the 1998 featuring.

At first sight this seems daunting – the 1000^{th} digit after the decimal point of $\sqrt{N}$! But perhaps 1000^{th} has been chosen because it is close to 999, which is half of 1998 ?

Let's try a simpler example. Take $N = 1111$ with only 4 digits so the answer can easily be checked on a hand calculator. This is $0 \cdot 1111 \times 10^4$. It looks like 1/9 without the infinitely recurring tail of 1s. If it were exactly 1/9, $\sqrt{N}$ would be 1/3, and every digit after the decimal point would be a 3.

We need to work out a correction term. Write

$$N = 10^4 \left(\frac{1}{9} - \frac{10^{-4}}{9} \right) \;=\; \frac{10^4}{9} \left(1 - 10^{-4} \right).$$

By the binomial theorem, the square root of this is approximately

$$\frac{10^2}{3} \left(1 - \frac{10^{-4}}{2} - \frac{10^{-8}}{8} - \ldots \right) \;=\; 100(0 \cdot 3333333\ldots \;- 0 \cdot 00005) \;=\; 33 \cdot 331666\ldots.$$

So $\sqrt{1111}$ is $33 \cdot 331666..$, from which we can pick out the digits of interest. The $(4/2+1)^{th}$ digit after the decimal point is a 1. If you repeat this with $N = 111111$, you get $\sqrt{N}$ is $333 \cdot 3331666..$ and the $6/2+1$ digit after the decimal point is a 1.

Now return to the Putnam question.

$$N = 10^{1998} \left(\frac{1}{9} - \frac{10^{-1998}}{9} \right) \;=\; \frac{10^{1998}}{9} \left(1 - 10^{-1998} \right).$$

$$\sqrt{N} \;\approx\; \frac{10^{999}}{3} \left(1 - \frac{10^{-1998}}{2} - \ldots \right).$$

The digit in place $1998/2 + 1$ after the decimal point is a 1.

This all works, of course, because the number of digits is even. It does not work for an odd number like 7; $\sqrt{(1111111)} = 1054 \cdot 09250068....$

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