**Q18 : Let $a$ and $b$ be positive integers. Show that**

$$\frac{(a+b)!}{(a+b)^{a+b}} \;<\; \frac{a!}{a^a}\frac{b!}{b^b}.$$

This was problem B-2 in the 2004 Putnam mathematical competition. (They used $m$ and $n$ instead.)

I have written this account as a case study in how we go about solving problems. These Putnam problems have usually not arisen naturally in the course of some research or application of maths, but are contrived with some witty 'trick' about them. This is a case in point. If you recognise the key, it opens the problem quickly and smoothly, like a well oiled lock. However, if you can't see the key, you merely mangle the lock in your attempt to pick it open with a stick or a nail, expending much effort to no avail. So my account of this problem is really a comment on how to tackle problems in general.

I failed to spot the key straight away and was feeling my way forwards by considering special cases when, in a chance conversation with Dr. Catherine Powell of Manchester University, she pointed out a similarity to a well known piece of elementary maths. She had recently been thinking of a similiar formula in connection with her own research. Once we saw this connection, the answer was blindingly obvious and immediately the problem became unlocked, in one line of maths!

In his classic book 'How to Solve It', George Polya discusses strategies and frames of mind for tackling maths problems. He urges us to ask 'Do you know a related problem?', 'Do you know a problem having the same unknown as the given problem?' So thanks to Dr Powell for demonstrating the power of Polya's guidance.

The challenge in this problem is seen on considering the two inequalities implied by the numerators and the denominators of these expressions. Suppose that $a \geq b$.

1. Numerators.

$$(a+b)! \;=\; (a+b)(a+b-1)\ldots(b+2)(b+1).b! \;>\; a(a-1)(a-2)\ldots 2.1.b! \;=\; a!b!$$

2. Denominators.

$$(a+b)^{a+b} \;=\; (a+b)^a.(a+b)^b \;>\; a^a.b^b\ .$$

Because both inequalities have the same sense, they tell us nothing about the ratio as posed in the question.

## 1. Key to the problem

We have the binomial $(a+b)^{a+b}$ so it is natural to think of its expansion by the binomial theorem. We also have a factorial $(a+b)!$ and the product $a!\,b!$ of two factorials. That rings a bell. The number of combinations of $n$ object taken $r$ at a time is

$${}^nC_r \;=\; \binom{n}{r} \;=\; \frac{n!}{r!\,(n-r)!}.$$

where $n!/r!$ is the truncated factorial $n(n-1)(n-2)\dots(r+1)$. These combinations feature as the coefficients in binomial expansions. So rearrange the given inequality to show that

$$\frac{(a+b)!}{a!\,b!}a^a\,b^b \;=\; \binom{a+b}{a}\,a^a\,b^b \;=\; \binom{a+b}{b}\,a^a\,b^b \;<\; (a+b)^{a+b}. \qquad 1)$$

The expansion of the right side is

$$a^{a+b} + \binom{a+b}{1}\,a^{a+b-1}\,b + \binom{a+b}{2}\,a^{a+b-2}\,b^2 + \cdots + \binom{a+b}{b}a^a\,b^b + \cdots + b^{a+b}.$$

The left side of Eq 1 is merely one term in this binomial expansion. All terms are positive, so this one term must be less than the sum. Problem solved, and very neatly too.

## 2. Without the key

The following notes give my thoughts before the elegant and definitive proof above. To gain some feel for the quantities involved and explore possible routes to solution, I considered some special cases. This is often a reasonable place to start, though in this case the lines of enquiry below were becoming increasingly complex and unsatisfactory.

**Case 1 :** $b = 1$. I took the inequality in the form given.

$$\frac{(a+1)!}{(a+1)^{a+1}} \;=\; \frac{a+1}{a+1}.\frac{a}{a+1}.\frac{a-1}{a+1}\cdots\frac{2}{a+1}.\frac{1}{a+1}$$

$$\frac{a!}{a^a} \;=\; \frac{a}{a}.\frac{a-1}{a}\cdots\frac{2}{a}.\frac{1}{a}.$$

The product in the first row above has $a+1$ factors, but the first equals 1. Apart from this, both rows show $a$ fractions in strictly decreasing order. Every fraction in the upper row has the same numerator but a larger denominator than the corresponding fraction in the lower row. Hence the product in the upper row is less than that in the lower row, meaning that the inequality in the question holds true for $b = 1$, any $a > 0$. I probably had in mind to attempt some proof by induction on $b$, in which case this could have been the base case.

**Case 2 :** $b = 2$. For this and Case 4, I recast the given inequality by removing the denominators. Define $\Delta$ by

$$\Delta \;=\; a!b!\,(a+b)^{a+b} - (a+b)!\,a^a b^b.$$

The challenge is therefore to show that $\Delta > 0$ for all $a \geq 1$, $b \geq 1$. With $b = 2$

$$\Delta \;=\; 2a!\,(a+2)^{a+2} - 4(a+2)!\,a^a.$$

Take out the factor $2(a+2)a! > 0$ and use the binomial theorem to expand $(a+2)^{a+1}$ :

$$\begin{aligned}\frac{\Delta}{2(a+2)a!} &= (a+2)^{a+1} - 2(a+1)\,a^a,\\ &= a^{a+1} + 2(a+1)a^a + \cdots + 2^{a+1} \;-\; 2a^{a+1} - 2a^a,\\ &= 3a^{a+1} + 2a^a + \cdots + 2^{a+1} \;-\; 2a^{a+1} - 2a^a,\\ &= a^{a+1} + \cdots + 2^{a+1}.\end{aligned}$$

All the terms implied by the dots $\dots$ are $> 0$, so $\Delta > 0$ as required. Note that this case, using the binomial expansion, comes fairly close to finding the key.

**Case 3 :** $a \gg 1,\, b \gg 1$. This case is just an interesting aside. When $a$ and $b$ are very large and positive, it is valid to use Stirling's asymptotic expansion of the factorial function in the form

$$\frac{N!}{N^N} \;\sim\; \sqrt{2\pi N}\exp\left(-N + \frac{\theta}{12N}\right), \qquad 0 < \theta < 1.$$

Hence

$$\frac{(a+b)!}{(a+b)^{a+b}} \;\sim\; \sqrt{2\pi(a+b)}\;\exp\left(-(a+b) + \frac{\theta}{12(a+b)}\right)$$

$$\frac{a!}{a^a}\frac{b!}{b^b} \;\sim\; 2\pi\sqrt{ab}\;\exp\left(-(a+b) + \frac{\theta_a}{12a} + \frac{\theta_b}{12b}\right)\;.$$

The two exponential tends asymptotically to each other, and $ab \gg a+b$, so the given inequality holds asymptotically.

**Case 4 :** $b = a$. Again I had induction in mind. Using $\Delta$ as defined for case 2

$$\Delta \;=\; (a!)^2\,(2a)^{2a} - (2a)!\,a^{2a} \;=\; (a!)^2\,4^a\,a^{2a} - (2a)!\,a^{2a}$$

A factor in this is $a!\,a^{2a}$. Introduce the symbol ${}^p\mathcal{T}_q$ to denote the truncated factorial :

$${}^p\mathcal{T}_q \;=\; \frac{p!}{q!} \;= p(p-1)(p-2)\dots(q+1), \qquad p > q, \qquad {}^p\mathcal{T}_p \;=\; 1.$$

${}^p\mathcal{T}_q$ has $p-q$ linear factors. Then

$$\frac{\Delta}{a!\,a^{2a}} \;=\; a!\,4^a \;-\; {}^{2a}\mathcal{T}_a. \qquad 2)$$

Special cases are : for $a = 1$, the right side is $4 - 2 = 2$, and for $a = 2$ it is $32 - 12 = 20$.

From these base cases we can prove that $\Delta > 0$ in Eq. 2 by induction on $a$. Assume the proposition that $4^a\,a! \;>\; {}^{2a}\mathcal{T}_a$ is true for some value of $a \geq 1$. The equivalent proposition for $a+1$ would compare

$$4^{a+1}\,(a+1)! \quad \text{with} \quad {}^{2(a+1)}\mathcal{T}_{a+1}.$$

$$\text{Now} \qquad 4^{a+1}\,(a+1)! \;=\; 4(a+1).(4^a\,a!) \;>\; 4(a+1)\,{}^{2a}\mathcal{T}_a \quad \text{by hypothesis,}$$

$$\text{whilst} \qquad {}^{2(a+1)}\mathcal{T}_{a+1} = \frac{(2a+2)(2a+1)}{a+1}\,{}^{2a}\mathcal{T}_a.$$

$$\text{Since} \qquad 4(a+1) \;=\; 4a+4 \;>\; 4a+2 \;=\; 2(2a+1),$$

the truth of the proposition for $a$ implies its truth for $a+1$. Hence this question's given inequality is true for any $b = a$.

I was considering how to generalise this analysis to the case $b \neq a$ when I met Dr. Powell. After her insight, there seemed little point taking it further.

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