Q19 : A dart, thrown at random, hits a square target. Find the probability that the point hit is nearer to the centre than to any edge.

This was problem B-1 in the 1989 Putnam mathematical competition. The question as stated gave two more conditions :

- Assume that any two parts of the target of equal area are equally likely to be hit,
- Express your answer in the form $\frac{a\sqrt{b}+c}{d}$ where a, b, c, d are integers.

With a uniform distribution, probability equates to fractional area. Therefore this is a straightforwards piece of geometry, to find the area of some inner figure co-centric with the given square. Figure 1 shows the upper right quadrant of the dart board (pale blue edge), centre O = (0,0), whose edges correspond with the lines $x = \pm 1$, $y = \pm 1$. P is a point equidistant from O and point Q on the closest edge.

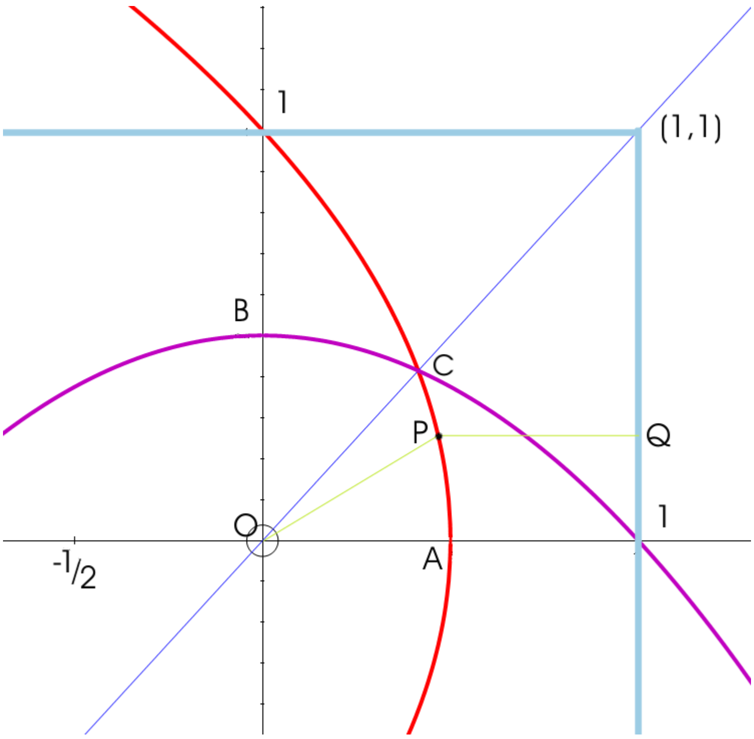

FIGURE 1 – First quadrant of the dart board showing lines equidistant from centre and edge.

It is clear that the locus of P is a parabola, because a parabola is defined as the locus of a point which moves such that its distances from a fixed point O and a fixed line (directrix, $x = 1$) are equal. Thus the red parabola through P defines the right hand edge of the required area. Similarly the purple parabola through B defines its upper edge. The probability required by the question is equal to the ratio of the area of the curvilinear triangle OBC to one eighth of the dart board.

P on the red parabola has co-ordinates (x_p, y_p) which satisfy

$$x_p^2 + y_p^2 = (1 - x_p)^2, \qquad \text{or} \qquad y_p^2 = 1 - 2x_p.$$

Similarly the purple parabola through B is $x_p^2 = 1 - 2y_p$. These two curves intersect at C on

$y = x$. Here

$$2x_c^2 \;=\; (1 - x_c)^2 \qquad \text{from which} \qquad x_c = y_c \;=\; \sqrt{2} - 1.$$

We'll work with the purple parabola since it is usual to integrate y as a function of x rather than *vice versa*. The area between arc BC and the x-axis is

$$\tfrac{1}{2}\int_0^{\sqrt{2}-1} (1 - x^2)\, dx \;=\; \tfrac{1}{2}\left[x - \frac{x^3}{3}\right]_0^{\sqrt{2}-1} \;=\; \tfrac{1}{3}\,(2 - \sqrt{2}\,).$$

To obtain area OBC we must subtract the area of the triangle under the diagonal OC. This is $x_c^2/2 = \frac{3}{2} - \sqrt{2}$, giving area OBC $= \frac{1}{6}(4\sqrt{2} - 5)$. Since 1/8 of the area of dart board is 1/2 unit, the required probability is

$$\tfrac{1}{3}(4\sqrt{2} - 5) \;=\; 21 \cdot 895\%.$$

This appears plausible since it is clear from Figure 1 that the fractional area and hence the probability is just less a quarter. As a check, I wrote a short computer program to simulate the dart being thrown, using the random number generator. Actual distances from centre and edges were calculated. Over four runs each of one million throws, the numbers of times the dart landed nearer the centre than the edge were respectively 219124, 218715, 218652 and 219164.

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