Q20 : Show that the curve $x^3 + 3xy + y^2 = 1$ contains only one set of three distinct points, A, B and C, which are vertices of an equilateral triangle, and find its area.

This was problem B1 in the 2006 Putnam mathematical competition. This is an example where considering a few special cases does reveal a fruitful line of enquiry.

We have to get some idea of the shape of this 2D curve, which is defined only implicitly. Clearly it is symmetric in x and y, so is mirrored about the line $y = x$. Already we might suspect that the required equilateral triangle will be symmetric about $y = x$.

A few points on the curve would be helpful. If $x = 0$, $y = 1$, and if $y = 0$, $x = 1$. Another obvious point to try is $x = \frac{1}{2}$, which gives $y = \frac{1}{2}$ also. But note that these three points lie on a straight line : $y = 1 - x$ as shown in Figure 1.

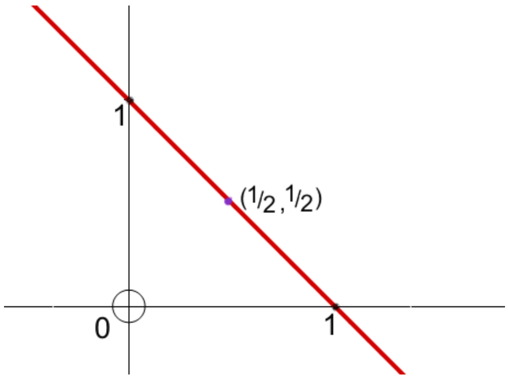

FIGURE 1 – Three points at which $x^3 + 3xy + y^3 = 1$.

So try $y = 1 - x$ in the given cubic :

$$x^3 + 3x(1-x) + (1 - 3x + 3x^2 - x^3) \;=\; 1.$$

This means that $x + y - 1$ must be a linear factor of $x^3 + 3xy + y^2 - 1$. Dividing by this gives the factorisation

$$x^3 + 3xy + y^2 - 1 \;=\; (x + y - 1)(x^2 - xy + x + y^2 + y + 1).$$

This will be zero if either factor is zero. The linear factor is zero everywhere along the straight line $y = 1 - x$, as already noted. Clearly this can only supply one side, two vertices of the equilateral triangle.

The quadatic factor can be solved for x by treating y as a constant :

$$x^2 + x(1-y) + (y^2 + y + 1) = 0 \quad \text{implies} \quad x = \tfrac{1}{2}[y - 1 \pm i\sqrt{3}(y + 1)].$$

This solution is complex everywhere except at the one point where the imaginary part is zero, namely $y + 1 = 0$. So P $= (-1, -1)$ is an isolated point on the cubic. At this stage the problem is essentially solved. The required equilateral triangle must join P to two points Q and R on $y = 1 - x$, as shown in Figure 2. The fact that there is only a single real point outside the line $y = 1 - x$ proves that this triangle is unique.

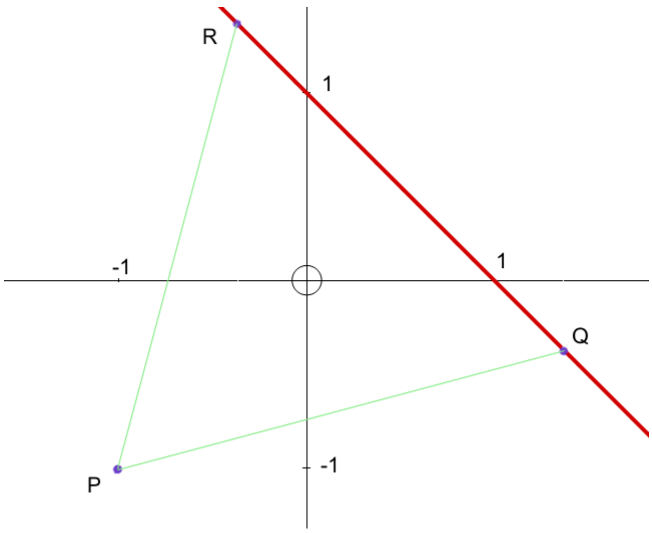

FIGURE 2 – The isolated point P at $(-1, -1)$ and the equilateral triangle PQR.

Let Q be $(x_0,\, 1 - x_0)$, making R $= (1 - x_0,\, x_0)$. By Pythagoras' theorem the square of the distance d between them is $d^2 = 2(2x_0 - 1)^2$. Similarly, the square of PQ is $2x_0^2 - 2x_0 + 5$. Equating these gives $x_0 = \frac{1}{2}(1 \pm \sqrt{3}\,)$. So

$$
\begin{aligned}
Q &= (\tfrac{1}{2}(1 + \sqrt{3}\,),\ \tfrac{1}{2}(1 - \sqrt{3}\,)) \\
R &= (\tfrac{1}{2}(1 - \sqrt{3}),\ \tfrac{1}{2}(1 + \sqrt{3}\,)) \\
d &= \sqrt{2}(2x_0 - 1) \;=\; \sqrt{6} \;\approx\; 2 \cdot 45.
\end{aligned}
$$

The area of the triangle is $\frac{d^2\sqrt{3}}{4} = \frac{3\sqrt{3}}{2} \approx 2 \cdot 6$ square units.

Taking this a little further, Figure 3 shows the surface $z = x^3 + 3xy + y^3$ with the points P, Q, R marked. Point P is a gentle local maximum. The constant $z = 1$ was chosen by the question master as the value at this maximum. The looped contour line is at $z = 0$.

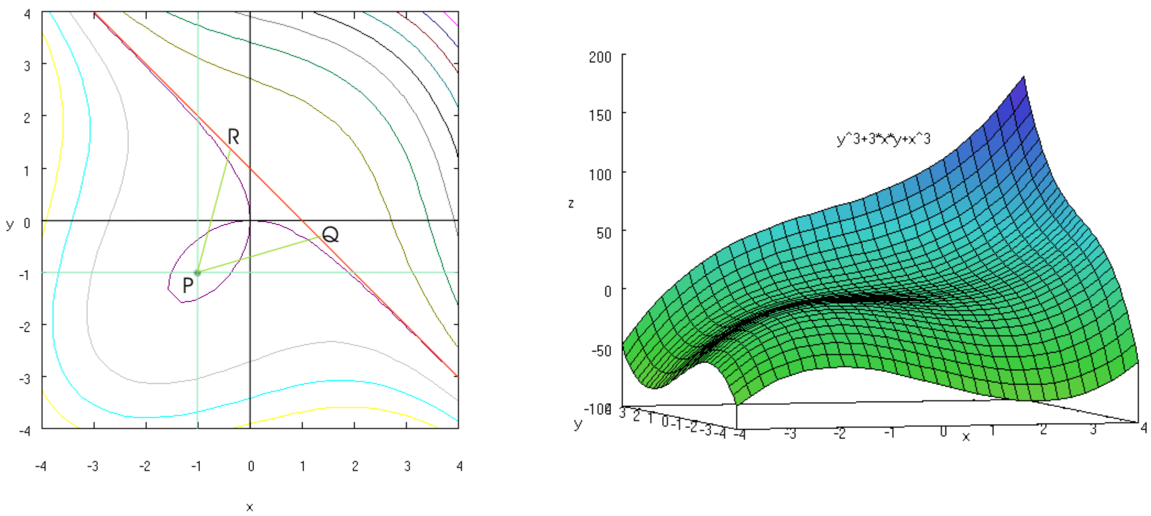

FIGURE 3 – Contour plot and perspective views of $x^3 + 3xy + y^3$ showing P as a local maximum.

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