Q23 : Prove that every non-zero coefficient of the Taylor series of

$$f(x) = (1 - x + x^2)\, e^x$$

about $x = 0$ is a rational number whose numerator (in lowest terms) is either 1 or a prime number.

This was question A1 in the 75th Putnam Maths Competition of December 2014.

The Taylor expansion of e^x about zero is very well known:

$$e^x \;=\; \sum_0^\infty \frac{x^n}{n!} \;=\; 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots .$$

The Taylor expansion of $f(x)$ is therefore

$$\begin{array}{cccccc} 1 & +x & +\frac{x^2}{2} & +\frac{x^3}{6} & +\frac{x^4}{24} & +\dots \\ & -x & -x^2 & -\frac{x^3}{2} & -\frac{x^4}{6} & -\dots \\ & & +x^2 & +x^3 & +\frac{x^4}{2} & +\dots \end{array}$$

$$= \; 1 + \frac{x^2}{2} + x^3\left(\frac{1}{6} - \frac{1}{2} + 1\right) + \dots$$

with the general term being

$$x^n\left(\frac{1}{(n-2)!} - \frac{1}{(n-1)!} + \frac{1}{n!}\right).$$

This simplifies to

$$x^n\,\frac{(n-1)^2}{n!} \;=\; x^n\,\frac{n-1}{(n-2)!\,n}.$$

So these are our coefficients. They are rational, as the question asks, though not in lowest terms. The question seems to be about how $n - 1$ and $(n - 2)!$ have common factors which cancel. (Clearly the n cannot cancel.)

If $n-1$ is prime, clearly it cannot cancel and so it remains in the numerator. This is one case of the numerator being a prime.

If $n-1$ is a product of primes, each of which occurs only once – that is, if $n-1 = p_1 p_2 \dots p_m$ – each of these primes is less than $n-1$ and so must occur as one of the ascending factors of $(n-2)!$. All these prime factors then cancel, leaving the numerator equal to 1. This is another case cited in the question.

The third case is where $n - 1$ is a power of a prime, or a product of such powers. Suppose that $n - 1 = p^k$, so the coefficient is

$$\frac{p^k}{(p^k + 1)\,(p^k - 1)!}.$$

In order for the numerator to be p, the other $k - 1$ multiples of p need to cancel, and to get 1, all k must cancel. This requires p to be a factor of the denominator k times. $(p^k - 1)!$ is formed from the ascending sequence of consecutive integers, every p^{th} one of which can cancel one of the p in the numerator. All will cancel if $p^k - 1 \geq kp$. In fact $p^k - 1 > kp$ for 2^3 and all larger values of p and k. To check this let us evaluate the first few non-zero coefficients of this form:

$$n - 1 = p^k = 2^2 : \qquad \frac{4}{5.3!} = \frac{2}{15}$$

$$2^3 : \qquad \frac{8}{9.7!} = \frac{1}{9.1.3.2.5.3.7}$$

$$2^4 : \qquad \frac{16}{17.15!} = \frac{1}{17.1.3.2.5.3.7.4.9.10.11.12.13.14.15}$$

$$3^2 : \qquad \frac{9}{10.8!} = \frac{1}{10.2.4.5.2.7.8}$$

In every case except 2^2 the denominator is sufficiently rich in factors equal to p that there is always full cancellation.

To summarise, for all prime values of $n - 1$ the numerator is $n - 1$. For all non-prime values of $n - 1$ except 4 have coefficients equal to 1. For $n - 1 = 4$ the numerator is 2.

I do not find this a very interesting question.

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