Q5: Which positive integers can be expressed as the sum of three or more consecutive positive integers?

Before we start, note the trivial cases that no even integer can be expressed as the sum of 2 consecutive numbers, but every positive odd integer can : $2m + 1 = m + (m+1)$. Twice an odd integer is then the sum of 4 terms: $2(2m + 1) = (m-1) + m + (m+1) + (m+2)$.

1. Which integers cannot be so expressed?

Suppose integer q is the sum of the k+1 integers n, $(n+1)$, $(n+2)$, ... , $(n+k)$. Then

$$q = \sum_0^k n + \sum_0^k k = n(k+1) + \tfrac{1}{2} k(k+1) = (k+1)(n + \tfrac{1}{2} k).$$

If k is even, $k = 2m$, say, then $q = (2m+1)(n+m)$.
If k is odd, $k = 2m+1$ say, $q = (2m+2)(n + m + \tfrac{1}{2}) = (m+1)(2n + 2m + 1)$.
In both cases q has at least one odd factor. Hence q cannot be a power of 2.

If q is prime, one of the factors must be 1 and the other must be q. Hence $m = 0$.
The case $m = 0$ for k even corresponds to the trivial case of only one term, $n = q$.
The case $m = 0$ for k odd gives $q = 2n+1$ and so limits q to being the sum of only two consecutive numbers, n and n+1: *e.g* 17 = 8 + 9. We conclude that no prime can be expressed as the sum of 3 or more consecutive positive integers.

So q cannot be a prime or a power of 2. These expressions impose no other constraints on q.

2. Negative integers

If we change the rules to allow some negative integers, primes can be expressed as sums of three or more consecutives. As an example,

$$17 = -7 - 6 - 5 \,.....\, - 1 + 0 + 1 + \,.....\, + 7 + 8 + 9.$$

The partial sum –7 – 6 – 5 ..... – 1 + 0 + 1 + ... + 7 = 0 is, of course, just padding to increase the number of terms above 2. A mathematical cheat!

3. An Algorithm

Here is an algorithm for determining n and k for any chosen sum q :

1) Select q. No solution in positive integers is possible if q is a power of 2. If q is prime, the only non-trivial solution is $q = (q-1)/2 + (q+1)/2$.

2) Select an odd divisor of q. We now have two choices:

3) Choice 1 : k is even
   a) The odd divisor is $k + 1$. Determine $k/2$.
   b) Divide q by $k + 1$. Call this j.
   c) Determine $n = j - k/2$. If $n < 0$, select another odd divisor and loop back to stage a).
   d) The required sum is $q = n + (n+1) + \ldots\ldots + (n+k)$.

4) Choice 2 : k is odd.

a) The odd divisor is $2(n+m)+1$.
b) Divide q by this odd divisor. Call the quotient h.
c) $h = m + 1$. Determine m and hence $k = 2m + 1$.
d) Find n from the divisor $2(n+m) + 1$ and m.
e) The required sequence is again $q = n + (n+1) + ... + (n+k)$

Example. Find sequences of consecutive integers summing to $q = 4922$.

Note that $4922 = 2 \times 23 \times 107$. The odd divisors are 23 and 107.

Case i) k even. Select the odd divisor to be 23, so $k/2 = 11$.
$j = 214$.
$n = 214 – 11 = 203$.
Thus 4922 = 203 + 204 + 205 +.....+ (203+22) (23 terms)

Case ii) k odd. With 23 as the odd divisor, $n + m = 11$.
$h = 214$
$m = 213$, $k = 427$
$n = –202$
Thus 4922 = –202 – 201 – 200 – …. + 225 (428 terms)
This is just case i) padded with the zero sum –202 – 201 – …. + 202.

Case iii) k even. This time select the odd divisor to be 107, so $k/2 = 53$.
$j = 46$.
$n = 46 – 53 = – 7$. Either reject this option or accept non-negative integers.
Thus 4922 = – 7 – 6 – 5 .... + 99 (107 terms)
Essentially in these last two cases there are simply too many terms for their sum to be 4922 without some being negative.

Case iv) k odd. Select the odd divisor to be 107, so $n + m = 53$.
$h = 46$
$m = 45$, $k = 91$
$n = 53 – 45 = 8$
Thus 4922 = 8 + 9 + 10 +.....+ 99 (92 terms).

So we have achieved two valid sequences of consecutive positive integers adding up to 4922.

4. Two sequences of terms

The examples show that 4922 = 203+204+.....+225 (A: 23 terms) and also 8+9+.....+99 (B: 92 terms). Are these two related? Note that $92 = 23\times4$. We can convert the longer sequence, B, into the shorter one, A, by these steps:

1) Add the first block of 23 terms of B termwise to the last block of 23 terms of B to produce 85, 87, 89 ... 127, 129.
2) Add the second block of 23 terms of B termwise to the third block of B to obtain 85, 87, 89 ... 127, 129 again.
3) Add these two derived blocks termwise to produce sequence C: 170, 174, 178, ... 250, 254, 258. The sum of these 23 terms is, of course, still 4922.
4) We now wish so collapse sequence C, which steps by 4, into one which steps by 1, to obtain consecutive integers.

5) The ‘padding’ between every two adjacent integers in C is 3 so with 22 steps the total padding is 66 which we split 33 to each side of the middle term. Strip out this padding as follows 170 + 33 = 203, 174 + 30 = 204, 178 + 27 = 205 .... 250 – 27 = 223, 254 – 30 = 224, 258 – 33 = 225 and so arrive at sequence A.

5. A Proof

The above is an elaborated version of how I presented the answer in 2009. In 2017 a friend, Gordon Robson, wrote to say that he had been looking at answering the question using a computer numerical search, as part of learning a new programming language. He found that *all* numbers up to 100,000 except primes and powers of 2 could be expressed as a sum of positive integers. Here is my explanation for this interesting result.

Perhaps we should not be surprised at this because the only restrictions on q are that it should not be prime or a power of 2.

We are representing q by adding $k + 1$ consecutive integers, $k \geq 2$, starting with $n \geq 1$. Then $q = n(k+1) + ½\, k(k+1)$, as found in §1 above. If k is even (so $k/2$ is an integer, m, say), this is $(2m+1)(n+m)$. Consider all the sequences of terms indexed by m, each generated by n as it increases from 1. The first few are

$m = 1$: 6, 9, 12, 15 .... stepping by 3,
$m = 2$: 15, 20, 25, 30, ... stepping by 5,
$m = 3$: 28, 35, 42, 49, ... stepping by 7,
$m = 4$: 45, 54, 63, 72, ... stepping by 9,
$m = 5$: 66, 77, 88, 99, ... stepping by 11,
$m = 6$: 91, 104, 117, ... stepping by 13, etc.

We claim that *every* integer q which is not prime or a power of 2 lies within at least two of these sequences, one corresponding to each prime factor of q. The reason is that the $m = 1$ sequence contains all integers divisible by 3, the $m = 2$ one all those divisible by 5 (though except 10 -- I discuss these ‘missing’ integers at the beginning of each sequence in §6 below), and so on for every odd prime 7, 11, 13, etc. Since q, non-prime, must have at least two prime factors, it must occur in at least the two respective sequences.

In particular, if p is the smallest odd prime factor of q, then q must lie in the sequence for $p = 2m + 1$ with p consecutive terms. We saw this above in the Example case 1 where $q = 4922$ and $p = 23$. The shortest sequence of consecutive positive integers which add to q has p terms in it, where p is the smallest odd prime factor of q.

To illustrate that there will be one sequence for each odd prime factor, take the example of 7021 = 7×17×59. It can be represented as

1000 + 1001 + ... + 1005 +1006 (7 terms), or
405 + 406 + ... + 420 + 421 (17 terms), or
90 + 91 + ... + 147 + 148 (59 terms).

Moreover, it is possible to expand each of these sequences to a multiple of 7, 17 or 59 terms by reversing the process used in §4. For instance, a sequence with 14 terms is

495 + 496 + ... + 507 + 508

and one with 34 terms is

190 + 191 + ... + 222 + 223.

6. Below-threshold integers

These is a loose end to tie up. The claim in §5 – that q must occur in at least the two sequences for which $2m+1$ is one of its prime factors – must be qualified because for each sequence there is a threshold below which integers q are not included. The below-threshold integers have the form $q = (2m + 1)(m + 1 - j)$ for $j = 1$ to $m - 1$. The first few are

$m = 2$: 10 = 1+2+3+4			
$m = 3$: 14 = 2+3+4+5	21 = 6+7+8		
$m = 4$: 18 = 5+6+7	27 = 8+9+10	36 = 11+12+13	
$m = 5$: 22 = 4+5+6+7	33 = 10+11+12	44 = 2+3+...+8+9	55 = 9+10+11+12+13.
$m = 6$: 26 = 5+6+7+8	39, 78 multiples of 3.	52 = 3+4+...+9+10	65 multiple of 5.

These examples show that the longest sequences are required for integers of the form $2^s p$ where p is the prime $2m+1$ and $q < (2m+1)(m+1)$. Two cases are

$248 = 2^3 \times 31 = 8+9+ .. +22+23$, (16 = 2^4 terms centred around 15·5 = 31/2)

$944 = 2^4 \times 59 = 14+15+ ... 44+45$ (32 = 2^5 terms centred around 29·5 = 59/2)

These are an extension of the fact noted at the beginning of this answer that twice an odd integer is the sum of 4 terms: $2(2m+1) = (m-1) + m + (m+1) + (m+2)$. Consider that

31 = 15 + 16

62 = 14 + (15+16) + 17

124 = 12 + 13 + (14+15+16+17) + 18 + 19

248 = 8 + 9 + 10 + 11 + (124) + 20 + 21 + 22 + 23

This construction shows how all below-threshold cases can be written as the sum of a consecutive sequence.

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