1 The definition and properties of the exponential and logarithm functions.

In an idle moment I tried to remember how the logarithm function was defined and how its relation to its inverse, the exponential, is developed. I know that this was all worked out in the late 17th century, but it does no harm to go back over these fundamentals. I looked in the classic book 'A Course in Pure Mathematics' by G. H. Hardy, and he first defines the logarithm in terms of the integral of $1/x$. However at school (many years ago) we had the exponential function defined first, and as the function which is everywhere equal to its own derivative. The logarithm was obtained later. In this short note I have retraced this topic, without looking back at the textbooks.

1.1 A power series for the exponential function

Take this function, $E(x)$, to be defined as that function of x which is everywhere equal to $dE(x)/dx$. I take x to be any real number and hope that there are no obstacles to extending it to any complex number. I further assume that $E(x)$ can be represented as a power series in x about $x = 0$ with the form

$$E(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \;....\; + a_k x^k \;..... \tag{1}$$

For the moment I leave open whether this is a finite or infinite series. If infinite, I intend to treat it with the same panache as Euler used to do and ignore the analytical exactitudes. Differentiating

$$\frac{dE(x)}{dx} = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \;....\; + k a_k x^{k-1} + (k+1)a_{k+1} x^k + ... \tag{2}$$

For this to hold for all x the coefficients of corresponding powers must be equal. Thus

$$a_1 = a_0, \quad a_2 = \frac{a_1}{2}, \quad a_3 = \frac{a_2}{3}, \quad ...., \quad a_{k+1} = \frac{a_k}{k+1}\,. \tag{3}$$

Once a_0 is given a value, all the other coefficients are determined. Nothing is lost if $a_0 = 1$. The coefficients are therefore

$$1, \quad \frac{1}{2}, \quad \frac{1}{2.3}, \quad \frac{1}{2.3.4.}, \quad ...$$

and the exponential function is represented by the very famous formula

$$E(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + ... + \frac{x^k}{k!} = \sum_{k=0}^{\infty} \frac{x^k}{k!}\,. \tag{4}$$

We need some justification for taking the series to infinity. The ratio of adjacent terms is

$$\frac{x^{k+1}}{(k+1)!} \cdot \frac{k!}{x^k} = \frac{x}{k}\,.$$

For any finite x this can be made arbitrarily small by taking k sufficiently large, so the series will converge by the ratio test. For complex numbers the ratio must be read as $|x|/k$.

Two special values are $E(0) = 1$, and $E(1) = \sum_{k=0}^{\infty} \frac{1}{k!}$. This is just a constant with a value close to $2 \cdot 72$ which the world calls 'e'.

1.2 Exponents and the functional equation

We need now to find out what this $E(x)$ function can do, so let us examine the product of $E(x)$ with $E(y)$. From the definition Eq 4

$$
\begin{array}{rclcccccccccccc}
 & & 1 & + & x & + & \frac{x^2}{2!} & + & \frac{x^3}{3!} & \ldots & + & \frac{x^k}{k!} & + & \ldots \\
 & & y & + & xy & + & \frac{x^2 y}{2!} & + & \frac{x^3 y}{3!} & \ldots & + & \frac{x^k y}{k!} & + & \ldots \\
 & & \frac{y^2}{2!} & + & \frac{xy^2}{2!} & + & \frac{x^2y^2}{2!.2!} & + & \frac{x^3y^2}{2!.3!} & \ldots & + & \frac{x^ky^2}{2!.k!} & + & \ldots \\
E(x).E(y) & = & \frac{y^3}{3!} & + & \frac{xy^3}{2!.3!} & + & \frac{x^2y^3}{2!.3!} & + & \frac{x^3y^3}{3!.3!} & \ldots & + & \frac{x^ky^3}{3!.k!} & + & \ldots \\
 & & \ldots. & & & & & & & & & & & \\
 & & \frac{y^k}{k!} & + & \frac{xy^k}{k!} & + & \frac{x^2y^k}{2!k!} & + & \frac{x^3y^k}{3!k!} & \ldots & & & + & \ldots \\
 & & \ldots. & & & & & & & & & & &
\end{array}
$$

You will notice the diagonal colouring. Collecting the terms in blue, green and red we have respectively

$$
\begin{gathered}
x + y \\
\frac{1}{2!}(x+y)^2 \\
\frac{1}{3!}(x+y)^3
\end{gathered}
$$

and the general diagonal will contain

$$
\frac{x^k}{k!} + \frac{x^{k-1}y}{(k-1)!} + \frac{x^{k-2}y^2}{2!.(k-2)!} + \frac{x^{k-3}y^3}{3!.(k-3)!} + \ldots + \frac{x^2y^{k-2}}{2!.(k-2)!} + \frac{xy^{k-1}}{(k-1)!} + \frac{y^k}{k!}
$$

$$
= \frac{1}{k!}\left(x^k + kx^{k-1}y + \frac{k(k-1)}{2!}x^{k-2}y^2 + \frac{k!}{3!(k-3)!}x^{k-3}y^3 + \ldots + k(k-1)x^2y^{k-2} + kxy^{k-1} + y^k\right).
$$

The infinite series in absolutely convergent so it is safe to reorder the terms in this way. You will recall the binomial expansion of $(x+y)^k$ with general term $x^m y^{k-m}\frac{k!}{m!(k-m)!}$. We obtain

$$
\begin{aligned}
E(x).E(y) &= 1 + (x+y) + \frac{1}{2!}(x+y)^2 + \frac{1}{3!}(x+y)^3 + \ldots + \frac{1}{k!}(x+y)^k + \ldots \\
&= E(x+y)\,. \qquad (5)
\end{aligned}
$$

This is called the functional equation and tells us that $E(x)$ shares the property with powers of any constant a that indices add when powers are multiplied: $a^2.a^5 = a^{2+5}$ and in general $a^x a^y = a^{x+y}$. This is also consistent with E(0)=1. It appears that $E(x)$ behaves as a^x for some constant a. Since $a^1 = a$ and $e^1 = e$, we can take $a = e$ and conclude that $E(x) = e^x$. Moreover, setting $y = -x$, $E(x-x) = 1$ so $E(-x) = 1/E(x)$, again consistent with $e^{-x} = 1/e^x$. $E(x)$ is usually notated as $\exp(x)$.

1.3 The logarithm as inverse function

The next stage is to investigate the properties of the inverse of $E(x)$. I will denote this by $L(u)$ where by definition

$$L(y) = E^{-1}(y) \;\text{ meaning that }\; L(E(x)) = x\,. \tag{6}$$

The functional equation for the exponential will require that $L(e^x e^y) = x + y$.

How to discover the structure of $L(u)$? Since $E(x)$ is known as an infinite power series, we could look for $L(u)$ also as a power series. The Handbook of Mathematical Functions compiled by Abramowitz and Stegun has at §3.6.25 formulae for reversion of series. To quote this great compendium of mathematical facts

Given $y = ax + bx^2 + cx^3 + dx^4 + ex^5 + fx^6 + gx^7 + ..$
then $x = Ay + By^2 + Cy^3 + Dy^4 + ...$
where
$aA = 1$
$a^3B = -b$
$a^5C = 2b^2 - ac$
$a^7D = 5abc - a^2d - 5b^3$
$a^9E = 6a^2bd + 3a^2c^2 + 14b^4 - a^3e - 21ab^2c$
plus equations for the next two coefficients.

In other words $x = Ax + (Ab + B)x^2 + (Ac + 2bB + C)x^3 + ...$ So $A = 1$. To match the other coefficients to Eq 4 where $b = 1/2$, $c = 1/6$, $d = 1/24$, $e = 1/120$ requires

$$B = -\frac{1}{2}, \quad C = \frac{1}{3}, \quad D = -\frac{1}{4}, \quad E = \frac{1}{5}\,.$$

This suggests that

$$L(y) \;=\; (y-1) - \frac{(y-1)^2}{2} + \frac{(y-1)^3}{3} - \frac{(y-1)^4}{4} + \frac{(y-1)^5}{5} - ..... + (-1)^y \frac{(y-1)^k}{k} + ... \tag{7a}$$

The $y-1$ argument allows for the exponential series starting $1+x+..$ whilst the series from Abramowitz and Stegun has no constant term. Extending this series beyond the first few terms is, of course, wishful conjecture. Since L(y) is known as the natural logarithm function, $\ln y$, we have established that for at least a finite number of terms

$$\ln(y+1) \;=\; y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \frac{y^5}{5} - \;.....\; + (-1)^{y-1}\frac{(y)^k}{k} + ... \tag{7b}$$

Immediately $\ln 1 = 0$, consistent with $e^0 = 1$. If y is negative, all terms are negative so $\ln u$ is negative if $u < 1$. The absolute ratio of consecutive terms is $\frac{ky}{k+1}$ which tends to y as $k \to \infty$, meaning that the series converges only for $|y| < 1$, and diverges for $y > 1$. For $y > 0$ it alternates and so grows only slowly. At the other extreme, as $y \to 1$ the series tends towards the harmonic series, which is known to diverge, so $\lim_{u\to -1} \ln u = -\infty$.

1.4 The logarithm as an integral

Eq 7b has the look of a Taylor series. The general form of $f(a+h)$ expanded about a for small h is

$$f(a+h) \;=\; f(a) + hf'(a) \;+\; \frac{h^2}{2!}.f''(a) \;+\; \frac{h^3}{3!}.f^{(3)}(a) \;+\; \frac{h^4}{4!}.f^{(4)}(a) + ...$$

Setting $a = 1$, $\ln 1 = 0$, $h = y$ gives

$$\ln(1+y) \;=\; y\frac{d\ln(1)}{dy} \;+\; \frac{h^2}{2!}\frac{d^2\ln(1)}{dy^2} \;+\; \frac{h^3}{3!}\frac{d^3\ln(1)}{dy^3} \;+\; \ldots$$

where the notation means that the derivatives are all evaluated at $y = 0$. We therefore know the values of the logarithm's derivative at 1. They are

$$\frac{d\ln u}{du}\Big|_{u=1} = 1, \quad \frac{d^2\ln u}{du^2}\Big|_{u=1} = -1, \quad \frac{d^3\ln u}{du^3}\Big|_{u=1} = 2!, \quad \frac{d^4\ln u}{du^4}\Big|_{u=1} = -3!, \quad \frac{d^5\ln u}{du^5}\Big|_{u=1} = 4!, \quad \ldots \tag{8}$$

Since $1 = 0! = 1!$, there are simple relations between consecutive derivatives: just multiply the previous by $-(n-1)$.

We naturally now think what function might give these values at 1 upon successive differentiation. The simple recursion rule suggests it is a simple function with a negative power in order to give the alternating + and − signs. Examine x^{-n}; its derivatives are

$$f' = -nx^{-(n+1)}, \quad f'' = n(n+1)x^{-(n+2)}, \quad f''' = -n(n+1)(n+2)x^{-(n+3)}, \quad f^{(4)} = n(n+1)(n+2)(n+3)x^{-(n+4)}\,.$$

These fit exactly to Eq 8 if $n = 1$, that is, if

$$\frac{d\ln u}{du} \;=\; \frac{1}{x} \text{ at } u = 1 \tag{9a}$$

and lead us to suspect that for general u

$$\ln u \;=\; \int_c^u \frac{1}{t}\,dt. \tag{9b}$$

The constant c at the lower limit is chosen so that $\ln c = 0$, which means from Eq 7a, b that $c = 1$. This is conjectural. Confidence would be increased if it could be shown that this definition implies that $\ln(ab) = \ln a + \ln b$, which is the logarithmic version of the functional equation.

1.5 Functional equation for the logarithm

Since Hardy starts with Eq 9b as his definition of the logarithm, he gives a beautifully crafted explanation of why $\ln(ab) = \ln a + \ln b$. I will not cheat by copying his book, but instead offer my own thinking.

Assume that Eq 9b is correct and change the variable from t to $r = t/a$ for some constant a. Then $dt = a\,dr$, when $t = 1, r = 1/a$ and when $t = u$, $r = u/a$.

$$\ln u \;=\; \int_1^u \frac{1}{t}\,dt \;=\; \int_{1/a}^{u/a} \frac{1}{r}\,dr\,.$$

Now set $u = a$ and invert the order of integration:

$$\ln a \;=\; \int_1^a \frac{1}{t}\,dt. \;=\; \int_{1/a}^1 \frac{1}{r}\,dr \;=\; -\int_1^{1/a} \frac{1}{r}\,dr \;=\; -\ln\left(\frac{1}{a}\right). \tag{10}$$

In a similar fashion and with the same substitution, $t = ar$,

$$\ln ab \;=\; \int_1^a \frac{1}{t}\,dt \;+\int_a^{ab} \frac{1}{t}\,dt. \;=\; \ln a + \int_1^b \frac{1}{r}\,dr \;=\; \ln a \;+\; \ln b. \tag{11}$$

This completes the analysis.

1.6 The calculation of logarithms

I do not know how the army of clerks in the 17th century compiled the first tables of logarithms for use in commerce and navigation. It would have involved days of tedious, painstaking hand calculations, writing each string of digits with a goose quill. Here is one approach which occurs to me. We have the representation of $\ln(1+y)$ as a power series valid for $|y| < 1$, its representation as an integral, and the functional equation with its corollaries:

$$\ln(ab) = \ln a + \ln b, \qquad \ln a = -\ln\frac{1}{a}, \qquad \ln a^b = b\ln a. \tag{12}$$

The last equation comes from generalising, say, $\ln a^4 = \ln(a\times a\times a\times a) = \ln a + \ln a + \ln a + \ln a = 4\ln a$. The series for $\ln(1+y)$ converges quickly when $|y|$ is small. The series' sum can be checked against numerical integration with, say, Simpson's 3/4 Rule, which integrates a cubic fitted through four equally spaced points spanning the interval from 1 to $1+y$. I find that Simpson's 3/8 Rule with only four points, one of which is 1, gives 5 figure accuracy for $0\cdot 7 < 1+y < 1$.

Consider these fractions which are all just less than 1:

$$\frac{4}{5}, \quad y = -\frac{1}{5}. \qquad \text{By series, 10 terms, } \ln\frac{4}{5} = -0\cdot 22314355,$$

$$\frac{5}{6}, \quad y = -\frac{1}{6}. \qquad \text{By series, 9 terms, } \ln\frac{5}{6} = -0\cdot 18232155,$$

$$\frac{8}{9}, \quad y = -\frac{1}{9}. \qquad \text{By series, 8 terms, } \ln\frac{8}{9} = -0\cdot 117783035.$$

These give three simultaneous equations in $\ln 2$, $\ln 3$ and $\ln 5$. In matrix form these are

$$\begin{pmatrix} 2 & 0 & 1 \\ 1 & -1 & 1 \\ 3 & -2 & 0 \end{pmatrix}\begin{pmatrix} \ln 2 \\ \ln 3 \\ \ln 5 \end{pmatrix} = \begin{pmatrix} -0\cdot 22314355 \\ -0\cdot 18232155 \\ -0\cdot 117783035 \end{pmatrix} \tag{13}$$

Inverting the square matrix gives $\ln 2 = 0\cdot 69314718$, $\ln 3 = 1\cdot 09861229$, $\ln 5 = 1\cdot 609437912$. The next log prime to calculate is $\ln 7$ and for this we might evaluate the series or use numerical integration to find $\ln(21/20) = 0\cdot 048790164 = \ln 3 + \ln 7 - 2\ln 2 - \ln 5$ from which $\ln 7 = 1\cdot 945910149$. From these four logarithms of primes it is straightforward to build a table of logs of 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 14, 15, 16, 18, 20, 21, 24, 25, 27, 28, 30, *etc.* and their reciprocals, and interpolate between them. Indeed, the table can be augmented by $\ln\sqrt{2} = \frac{1}{2}\ln 2$, etc. since $\sqrt{2} = 1\cdot 41421...$ and other square roots can be calculated algebraically by successive approximation on $(A+\delta)^2 \approx A^2 + 2A\delta$ for δ small. For $\sqrt{2} = \frac{1}{10}\sqrt{200}$ one could start with $A^2 = 14^2 = 196$.

I may have been fortunate in choosing the three fractions built from the primes 2, 3, 5, to solve for the logarithms of these integers at Eq 13. I have found it practical to use one small prime such as 2 or 3 with a close pair of larger primes separated by 2: for instance 2 with 11 and 13. Suitable fractions for these numbers are

$$\frac{11}{13}, \qquad \frac{13^4}{2.11^4}, \qquad \frac{11^2}{2^7}.$$

The solution gives $\ln 2 = 0\cdot 693147182$ which agrees exactly with the value from Eq 13. Also $\ln 11 = 2\cdot 39789528$ and $\ln 13 = 2\cdot 56494936$ which are correct to 8 decimal places. Getting the bit between my teeth I used 2, 17, 19 in these fractions

$$\frac{17}{19}, \qquad \frac{2^4}{17}, \qquad \frac{19^6}{2.17^6}$$

to confirm $\ln 2$ yet again and find $\ln 17 = 2\cdot 83321334$ and $\ln 19 = 2\cdot 94443898$. Since we know $\ln 2$ and $\ln 3$, $\ln 19$ can also be found from $3^6/(2\times 19^2) = 1\cdot 0097..$ to be $2\cdot 94443898$. Pressing on, $\ln 23$ can be obtained from $23^6/(3\times 19^6) = 1\cdot 0488..$ or even from the $2\times 3^5/23^3 = 0\cdot 9187..$ Clearly each and every prime added to the growing table of logs fills in many gaps and so increases the ability to interpolate intermediate values. If only they had had even mechanical calculators in Newton's time!

Of course, for practical calculations the pioneering users of logarithms wanted logs to base 10, not e. How were these to be calculated? The clue is in the formula for a 'power of a power': *e.g.* $(a^5)^3 = a^5.a^5.a^5 = a^{15} = a^3.a^3.a^3.a^3.a^3 = (a^3)^5$ and in general $(a^b)^c = a^{(bc)} = (a^c)^b$. So if

$$b = c^q \quad \text{and} \quad c = a^p, \quad b = a^{pq}.$$

In terms of logarithms these relations are

$$q = \log_c b, \quad p = \log_a c, \quad pq = \log_a b,$$

$$\text{so} \qquad \log_a c.\log_c b = \log_a b.$$

$$\log_{10} x = \log_{10} e.\log_e x \tag{14}$$

so we just need to know $\log_{10} e$. Setting $x = 10$, $1 = \log_{10} e.\log_e 10$ so $\log_{10} e = 1/(\log_e 2 + \log_e 5) = 1/2\cdot 302585 = 0\cdot 434294482$. In other words, log tables to base 10 are produced from ones to base e simply by scaling all values to about 43.4%.

1.7 Application to the binomial series

It is well known that the binomial theorem, which expands $(a+b)^n$, is a polynomial in a and b when n is a positive integer. The simplest cases are $(a+b)^2 = a^2+2ab+b^2$ and $(a+b)^3 = a^3+3a^2b+3ab^2+b^3$. The binomial coefficients for large n are well tabulated: ${}^nC_r = n!/[r!(n-r)!]$. The ratio $n!/(n-r)!$ is interpreted as $n(n-1)(n-2)(n-3)...(n+1-r)$. The expansion can in some cases be given an obvious meaning when the exponent is a fraction. Thus $(a+b)^{1/2}$ means the series which, when squared, is equal to $a+b$. Write

$$(a+b)^{\frac{1}{2}} = a^{\frac{1}{2}}(1+x)^{\frac{1}{2}} = a^{\frac{1}{2}}\left[1 + c_1x + c_2x^2 + c_3x^3 + ...\right], \qquad x = \frac{b}{a}.$$

Squaring both sides

$$a(1+x) = a[1 + 2c_1x + (c_1^2 + 2c_2)x^2 + 2(c_1c_2 + c_3)x^3 + (2c_1c_3 + c_2^2 + 2c_4)x^4 + ...] \tag{15}$$

so by equating coefficients

$$c_1 = \frac{1}{2}, \qquad c_2 = -\frac{1}{8}, \qquad c_3 = \frac{1}{16}, \qquad c_4 = -\frac{5}{128}, \qquad etc.$$

This is also the Taylor expansion of $(1+x)^{1/2}$ about $x = 0$. The coefficients are consistent with the formula

$$(1+x)^q = 1 + qx + \frac{q(q-1)}{2!}x^2 + \frac{q(q-1)(q-2)}{3!}x^3 + \frac{q(q-1)(q-2)(q-3)}{4!}x^4 + ... \tag{16}$$

where q is a fraction. These coefficients are clearly a generalisation of the binomial coefficients with the fraction q merely replacing the integer n. The series in Eq 15 is infinite and converges only if $|x| < 1$, two major differences from the exponent being an integer.

This introduction sets the scene for a further generalisation to cases where the exponent is an irrational or even transcendental number, such as $(1+x)^{\sqrt{2}}$ or $(1+x)^{\pi}$. What meaning is to be attached to such expressions, when there is no way of multiplying them to produce $1+x$ alone? Stepping back from this, what meaning is to be attached to $3^{\sqrt{2}}$ or 5^{π}? Numerical values can be found for both of these using logarithms. We have

$$\ln[3^{\sqrt{2}}] \ = \ \sqrt{2}\times\ln 3 \ = \ 1\cdot 55367... \quad \text{so} \quad 3^{\sqrt{2}} \ = \ \exp(1\cdot 55367...) \ = \ 4\cdot 728804....$$

Note that here we have stretched the functional equation for the logarithm, Eq 11. In Eq 12 $\ln a^b = b\ln a$ followed on the understanding that b is an integer. We are now claiming that for any b, $\ln a^b$ is *defined* by $b\ln a$. It is certainly consistent with fractional values of b such as

$$\ln a \ = \ \ln[a^{1/2}.a^{1/2}] \ = \ \ln a^{1/2}+\ln a^{1/2} \ = \ 2\ln a^{1/2}\,.$$

Suppose that $|x|<1$ and

$$(1+x)^{\alpha} \ = \ 1+c_1x+c_2x^2+c_3x^3+....$$

for *any* value of α. Then

$$\alpha\ln(1+x) \ = \ \alpha[x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{3}-...]$$

from Eq 7b. Inverting the logarithm

$$(1+x)^{\alpha} \ = \ \exp(\alpha[x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{3}-...])$$

Now use the series definition of the exponential from Eq 4 to expand this as a power series in x. The first few coefficients are

$$c_1=\alpha, \qquad c_2=\frac{1}{2}\alpha(\alpha-1), \qquad c_3=\frac{1}{6}\alpha(\alpha-1)(\alpha-2),$$

$$c_4=\frac{1}{24}\alpha(\alpha-1)(\alpha-2)(\alpha-3), \qquad c_5=\frac{1}{120}\alpha(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)\,.$$

We are not surprised but nevertheless relieved to see that these are entirely in agreement with Eq 16. Though it is strictly necessary to prove that this pattern continues to infinity, it seems highly likely that the binomial expansion Eq 16 holds for all exponents. It is quite a remarkable generalisation.

John Coffey, 2019, extended 2020